Handbook on geometry

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§1. Notations and conventions

Transformation (active transformation) of manifold $M \subset \mathbb{R}^n$ is one-to-one map $x \to x' = f(x)$. Each transformation f can be considered as *coordinate transformation* (passive transformation, transformation of coordinate system) in such a way that the old x and new ξ coordinates are related by $x = f(\xi)$. If not specified we distinguish active and passive transformations by using the decoration of the same symbol (x and x') for active and different symbol (x and ξ) for passive transformations. In most cases we treat transformations as active.

§2. Linear transformations

2.1. General formulas

Consider a linear space $\mathbb{R}^n$. We denote the coordinates of vector $x \in \mathbb{R}^n$ and matrix (operator) $A : \mathbb{R}^n \to \mathbb{R}^n$ by indexes below: x_i and A_{ij}.

Linear transformation is a linear map $x \to x' = Tx$, where T is a nondegenerate matrix. They compose a group $GL(n, \mathbb{R})$. *Translation* is a map $x \to x' = x + a$, where a is a vector. The corresponding group is denoted by $T(n, \mathbb{R})$. Combinations of linear transformations and translations form *affine transformations*, $x \to x' = Tx + a$. The affine group $Aff(n, \mathbb{R}) = GL(n, \mathbb{R}) \ltimes T(n, \mathbb{R})$ since $(T, a)(1, b)(T, a)^{-1} = (1, Tb)$. Any linear transformation of vectors (or linear part of affine transformation) induces the following transformation of matrices: $A \to A' = TAT^{-1}$.

Any basis $\{e^i\}$ transforms to basis $\varepsilon^i = Te^i$ so that for any $x \in \mathbb{R}^n$ we have $x = \sum_i x_i e^i = \sum_i \xi_i \varepsilon^i$ and thus $x_i = \sum_j T_{ij} \xi_j$ and $\varepsilon^i = \sum_j T_{ji} e^j$.

In Euclidean space the *orthogonal transformations* are defined as those preserving the scalar product: $(Ox, Oy) = (x, y)$. The corresponding group is denoted by $O(n)$ and it is the group of matrices O with $\det O = \pm 1$. *Rotation* R (proper) is the orthogonal transformation with $\det R = 1$, the corresponding group is denoted by $SO(n)$. Any orthogonal transformations is either rotation or composition of a rotation and *inversion* $(x \to -x)$ yielding $O(n) = SO(n) \times I$, where $I = \{1, -1\}$. Any rotation in $\mathbb{R}^n$ can be considered as commutative superposition of rotations in $[n/2]$ mutually orthogonal planes. The default matrix of rotation on angle α in plane xy

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

rotates all vectors in the direction from x to y that is clockwise if watching along the rotation axis in $\mathbb{R}^3$ (right hand rule). The group of *motions* of $\mathbb{R}^n$ is $IO(n, \mathbb{R}) = O(n) \ltimes T(n, \mathbb{R})$. *Scaling* is a linear transformation $x \to x' = \alpha x$, $\alpha > 0$. The combinations of motion and scaling form the group of *conformal* transformations.

To find the matrix of a given linear transformation use the following rule: j-th column of the matrix is the transformation of the unit vector directed along j-th axis.

2.2. Linear transformations in 3D

In $\mathbb{R}^3$ any rotation R is the rotation on an angle α around some axis $\boldsymbol{n} = (n_x, n_y, n_z) = (\theta, \phi)$. Its infinitesimal operator is

$$L = \begin{pmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{pmatrix},$$

so that

$$R(\alpha, \boldsymbol{n}) = e^{L\alpha} \equiv 1 + L\sin\alpha + L^2(1 - \cos\alpha).$$

In vector notations

$$L\boldsymbol{r} = \boldsymbol{n} \times \boldsymbol{r}, \qquad R\boldsymbol{r} = \boldsymbol{r}\cos\alpha + \boldsymbol{n}(\boldsymbol{n}\boldsymbol{r})(1 - \cos\alpha) + (\boldsymbol{n} \times \boldsymbol{r})\sin\alpha.$$

Note the identities:

$$R(\alpha, \boldsymbol{n})^{-1} = R(-\alpha, \boldsymbol{n}), \qquad R(\alpha, \boldsymbol{n}) = R_z(\phi)R_y(\theta)R_z(\alpha)R_y(-\theta)R_z(-\phi),$$

where R_x is the rotation around x-axis.

For a given rotation matrix R its axis and angle of rotation can be calculated as follows. The axis $\boldsymbol{n}$ is the eigenvector corresponding to eigenvalue 1 of the matrix R. Let $\boldsymbol{h}_1 + i\boldsymbol{h}_2$ be the eigenvector corresponding to eigenvalue $\lambda_1 + i\lambda_2$ and the triple $\{\boldsymbol{h}_1, -\boldsymbol{h}_2, \boldsymbol{n}\}$ is right-handed, then the angle $\alpha = \arctan(\lambda_1, \lambda_2)$. The Jordan form of R and the transformation matrix, defined by $R = TJT^{-1}$ (there is a basis where T is rotation), are

$$J = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} \boldsymbol{h}_1 & -\boldsymbol{h}_2 & \boldsymbol{n} \end{pmatrix} \equiv R_z(\phi)R_y(\theta).$$

Another description of rotations are provided by Euler's angles. There are different conventions concerning Euler's angles. The standard convention is "x-convention" with precession angle ϕ, nutation angle θ, and self-rotation angle ψ (see Fig. 1). The three basic vectors (ξ, η, ζ) are rotated on angle ϕ around ζ (at this moment it is z-axis), then it is rotated on angle θ around ξ (at this moment it is the ascending nodes line ☊), and finally it is rotated again around ζ on angle ψ. The resulting transformation matrix is given by

$$R(\phi, \theta, \psi) = R_z(\phi)R_x(\theta)R_z(\psi) = \begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\cos\phi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix},$$

so that $R^{-1}(\phi, \theta, \psi) = R(-\psi, -\theta, -\phi)$. Thus defined matrix R rotates x to ξ, y to η, z to ζ, i.e. $R\boldsymbol{e}_x = \Upsilon'$ in Fig. 1.

In quantum mechanics and spherical trigonometry a so called "y-convention" is used. The transformation

$$R(\alpha, \beta, \gamma) = R_z(\alpha)R_y(\beta)R_z(\gamma), \qquad R^{-1}(\alpha, \beta, \gamma) = R(-\gamma, -\beta, -\alpha)$$

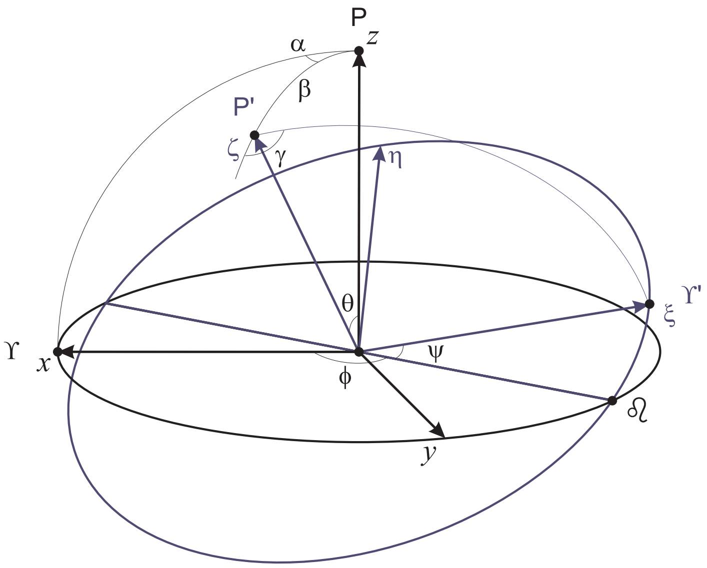

Figure 1: Euler's angles (ϕ, θ, ψ) and (α, β, γ).

is defined by its pole (β, α) (that is ζ-direction) and vernal equinox angle γ that is the angle between the projection of ζ onto xy plane and ξ (vernal equinox Υ' in Fig. 1). The relation between standard and spherical Euler's angle are as follows:

$$\alpha = \phi - \pi/2, \quad \beta = \theta, \quad \gamma = \psi + \pi/2.$$

The transformation from Euler's angles (α, β, γ) to rotational parameters (α', θ, ϕ) is given by

$$\cot\theta = \cot\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2}, \qquad \cos\frac{\alpha'}{2} = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}, \qquad \phi = \frac{\alpha-\gamma+\pi}{2},$$

and its inverse is

$$\sin\frac{\beta}{2} = \sin\theta\sin\frac{\alpha'}{2}, \qquad \alpha+\gamma = 2\arctan\left(\cos\theta\sin\frac{\alpha'}{2}, \cos\frac{\alpha'}{2}\right), \qquad \alpha-\gamma = 2\phi - \pi.$$

Note that the above transformations provide one to one mapping only in terms of generated rotation matrices.

Reflection in plane with normal vector $\boldsymbol{n}$ is given by matrix $-R(\pi, \boldsymbol{n})$, *mirror rotation* is given by $-R(\alpha + \pi, \boldsymbol{n})$.

§3. Coordinate systems

3.1. General formulas

Let a coordinate transformation $\boldsymbol{x} = f(\xi)$ is given, where $\boldsymbol{x} \in \mathbb{R}^n$. Cartesian indexes we denote by Latin letters: $i, j, \ldots$, and $\sum_i$ means $\sum_{i=1}^n$. New coordinates we write as ξ^α and the indexing is by Greek letters. We assume summation over dummy indexes, i.e. expression $A_{\alpha\beta}B^{\beta\gamma}$ means $\sum_{\beta=1}^n A_{\alpha\beta}B^{\beta\gamma}$.

Local basis is basis in tangent space defined in every non-singular point (det $\frac{\partial x}{\partial \xi} \neq 0$ or ∞). Two local bases are distinguished. The first is *natural basis*: $\boldsymbol{e}_\alpha = \frac{\partial \boldsymbol{x}}{\partial \xi^\alpha}$. *Metric tensor* is defined as symmetric positively definite matrix with elements

$$g_{\alpha\beta} = (\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta) \equiv \sum_i \frac{\partial x_i}{\partial \xi^\alpha}\frac{\partial x_i}{\partial \xi^\beta}.$$

Since $\mathrm{d}x_i = \frac{\partial x_i}{\partial \xi^\alpha}\,\mathrm{d}\xi^\alpha$, a Cartesian arc element in new coordinates is given by

$$\mathrm{d}s^2 \equiv \sum_i \mathrm{d}x_i^2 = g_{\alpha\beta}\,\mathrm{d}\xi^\alpha\,\mathrm{d}\xi^\beta.$$

Inverse metric tensor is denoted as $g^{\alpha\beta} = (g^{-1})_{\alpha\beta}$ so that $g^{\alpha\gamma} g_{\gamma\beta} = g_{\beta\gamma} g^{\gamma\alpha} = \delta^{\alpha}_{\beta}$. A *dual basis* is $\boldsymbol{e}^{\alpha} = g^{\alpha\beta}\boldsymbol{e}_{\beta}$, so that $g^{\alpha\beta} = (\boldsymbol{e}^{\alpha}, \boldsymbol{e}^{\beta})$. Vectors of the natural basis are parallel to coordinate curves, vectors of dual basis are perpendicular: $(\boldsymbol{e}^{\alpha}, \boldsymbol{e}_{\beta}) = \delta^{\alpha}_{\beta}$. The linear span of the natural basis is called *contravariant* vector space and the linear span of the dual basis is called *covariant* vector space.

3.2. Orthogonal coordinate systems

The coordinate system is *orthogonal* if its metric tensor is diagonal that is coordinate curves are orthogonal in each non-singular point. In orthogonal coordinate systems the default basis is so called *physical basis* (*unit vectors*), which is the normalized natural basis: $\boldsymbol{i}_{\alpha} = \sqrt{g_{\alpha\alpha}}\boldsymbol{e}_{\alpha}$. The notions covariant and contravariant are not used in this case, the vector space is linear span of the physical basis.

Differential operators:

$$\begin{aligned}
\boldsymbol{\nabla}\Phi &= \sum_{\alpha} \frac{1}{\sqrt{g_{\alpha\alpha}}} \frac{\partial \Phi}{\partial \xi^{\alpha}} \boldsymbol{i}_{\alpha}, \\
\boldsymbol{\nabla}\boldsymbol{A} &= \sum_{\alpha} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^{\alpha}} \left(\sqrt{\frac{g}{g_{\alpha\alpha}}} A_{\alpha} \right), \\
\Delta\Phi &= \sum_{\alpha} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^{\alpha}} \left(\sqrt{\frac{g}{g_{\alpha\alpha}}} \frac{\partial \Phi}{\partial \xi^{\alpha}} \right).
\end{aligned}$$

3.3. Cylindrical and polar coordinates

Cylindrical (in 2D *polar*) coordinates are defined by the transformations:

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

Differentials of coordinates transform as follows:

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\rho\sin\phi & 0 \\ \sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}\rho \\ \mathrm{d}\phi \\ \mathrm{d}z \end{pmatrix}.$$

Unit vectors and their derivatives:

$$\begin{pmatrix} \boldsymbol{i}_{\rho} \\ \boldsymbol{i}_{\phi} \\ \boldsymbol{i}_{z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \left(\frac{\partial \boldsymbol{i}_{\alpha}}{\partial x_j} \right) = \begin{pmatrix} 0 & \boldsymbol{e}_{\phi} & 0 \\ 0 & -\boldsymbol{e}_{\rho} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Metric tensor, Jacobian, and connectivity:

$$g_{\rho\rho} = 1,\ g_{\phi\phi} = \rho^2,\ g_{zz} = 1, \quad \sqrt{g} = \rho, \qquad \Gamma^{\rho}_{\phi\phi} = -\rho,\ \Gamma^{\phi}_{\rho\phi} = 1/\rho.$$

Differential operators:

$$\begin{aligned}
\boldsymbol{\nabla} f &= \left(\frac{\partial f}{\partial \rho}, \frac{1}{\rho}\frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial z} \right), \\
\boldsymbol{\nabla}\boldsymbol{A} &= \frac{1}{\rho}\frac{\partial(\rho A_{\rho})}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_{\phi}}{\partial \phi} + \frac{\partial A_z}{\partial z}, \\
\boldsymbol{\nabla}\times\boldsymbol{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_{\phi}}{\partial z}, \frac{\partial A_{\rho}}{\partial z} - \frac{\partial A_z}{\partial \rho}, \frac{1}{\rho}\frac{\partial(\rho A_{\phi})}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_{\rho}}{\partial \phi} \right), \\
\Delta f &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}.
\end{aligned}$$

3.4. Spherical coordinates

Spherical coordinates are defined by the transformations:

$$x = r\sin\theta\cos\phi, \quad y = r\cos\theta\sin\phi, \quad z = r\cos\theta.$$

Differentials of coordinates transform as follows:

$$\begin{pmatrix} \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta \\ \mathrm{d}\phi \end{pmatrix}.$$

Unit vectors and their derivatives:

$$\begin{pmatrix} \boldsymbol{i}_r \\ \boldsymbol{i}_\theta \\ \boldsymbol{i}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}, \qquad \left(\frac{\partial \boldsymbol{i}_\alpha}{\partial x_j}\right) = \begin{pmatrix} 0 & \boldsymbol{e}_\theta & \sin\theta\ \boldsymbol{e}_\phi \\ 0 & -\boldsymbol{e}_r & \cos\theta\ \boldsymbol{e}_\phi \\ 0 & 0 & -\sin\theta\ \boldsymbol{e}_r - \cos\theta\ \boldsymbol{e}_\theta \end{pmatrix}.$$

Metric tensor, Jacobian, and connectivity:

$$g_{rr} = 1,\ g_{\theta\theta} = r^2,\ g_{\phi\phi} = r^2\sin^2\theta, \quad \sqrt{g} = r^2\sin\theta,$$
$$\Gamma^r_{\theta\theta} = -r,\ \Gamma^r_{\phi\phi} = -r\sin^2\theta,\ \Gamma^\theta_{r\theta} = 1/r,\ \Gamma^\phi_{r\phi} = 1/r,\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta,\ \Gamma^\phi_{\theta\phi} = \cot\theta.$$

Differential operators:

$$\begin{aligned}
\boldsymbol{\nabla} f &= \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right), \\
\boldsymbol{\nabla}\boldsymbol{A} &= \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}, \\
\boldsymbol{\nabla}\times\boldsymbol{A} &= \left(\frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}, \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}, \frac{1}{r}\frac{\partial(rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right), \\
\Delta f &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.
\end{aligned}$$

3.5. Hyperspherical coordinates

Hyperspherical coordinates $(r, \theta_n, \theta_{n-1}, \ldots, \theta_1, \theta_0)$, where $r \geqslant 0$, $\theta_k \in (0, \pi)$, $k \geqslant 1$, $\theta_0 \in (0, 2\pi)$, are defined in $\mathbb{R}^{n+2}$ by

$$\begin{cases} x_1 = r\sin\theta_n \ldots \sin\theta_2\sin\theta_1\sin\theta_0, \\ x_2 = r\sin\theta_n \ldots \sin\theta_2\sin\theta_1\cos\theta_0, \\ x_3 = r\sin\theta_n \ldots \sin\theta_2\cos\theta_1, \\ \ldots \\ x_k = r\sin\theta_n \ldots \sin\theta_{k-1}\cos\theta_{k-2}, \\ \ldots \\ x_{n+1} = r\sin\theta_n\cos\theta_{n-1}, \\ x_{n+2} = r\cos\theta_n. \end{cases}$$

Inverse transformations are

$$r = \sqrt{x_1^2 + \ldots + x_{n+2}^2}, \quad \tan\theta_k = \frac{1}{x_{k+2}}\sqrt{x_1^2 + \ldots + x_{k+1}^2}.$$

Differentials of coordinates transform as follows:

$$\begin{pmatrix} \mathrm{d}x_1 \\ \mathrm{d}x_2 \\ \mathrm{d}x_3 \\ \ldots \\ \mathrm{d}x_{n+1} \\ \mathrm{d}x_{n+2} \end{pmatrix} = \begin{pmatrix} x_1/r & x_1\cot\theta_n & \ldots & x_1\cot\theta_2 & x_1\cot\theta_1 & x_1\cot\theta_0 \\ x_2/r & x_2\cot\theta_n & \ldots & x_2\cot\theta_2 & x_2\cot\theta_1 & -x_2\tan\theta_0 \\ x_3/r & x_3\cot\theta_n & \ldots & x_3\cot\theta_2 & -x_2\tan\theta_1 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n+1}/r & x_{n-1}\cot\theta_n & -x_{n-1}\tan\theta_n & 0 & \ldots & \ldots \\ x_{n+2}/r & -x_{n+2}\tan\theta_n & 0 & 0 & \ldots & \ldots \end{pmatrix} \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\theta_n \\ \mathrm{d}\theta_{n-1} \\ \ldots \\ \mathrm{d}\theta_1 \\ \mathrm{d}\theta_0 \end{pmatrix}$$

Metric tensor and Jacobian:

$$g_{rr} = 1,\ g_{\theta_k\theta_k} = r^2 \sin^2\theta_n \dots \sin^2\theta_{k+1}, \qquad \sqrt{g} = r^{n+1}\sin^n\theta_n \dots \sin^2\theta_2 \sin\theta_1.$$

Hence the Laplacian is

$$\Delta = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \frac{\Delta_n}{r^2},$$

where Δ_n is the Laplacian on sphere:

$$\Delta_n = \frac{1}{\sin^n\theta_n}\frac{\partial}{\partial\theta_n}\left(\sin^n\theta_n\frac{\partial}{\partial\theta_n}\right) + \frac{\Delta_{n-1}}{\sin^2\theta_n}, \qquad \Delta_0 = \frac{\partial^2}{\partial\theta_0^2}.$$

Note that manifold Ω specified by fixing $r, \theta_n, \dots, \theta_k$ is the hypersphere in $\mathbb{R}^{k+1}$, hence

$$\int_{\mathbb{R}^{n+2}} f(r, \theta_n, \dots, \theta_k)\, \mathrm{d}x_1 \dots \mathrm{d}x_{n+2} = S_{k+1}\int_\Omega f(r, \theta_n, \dots, \theta_k)\, r^{n+1}\sin^n\theta_n \dots \sin^k\theta_k\, \mathrm{d}r\, \mathrm{d}\theta_n \dots \mathrm{d}\theta_k,$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$.

§4. Euclidean geometry

Circumsphere. Let $\{\boldsymbol{r}_0, \boldsymbol{r}_1, \dots, \boldsymbol{r}_n\}$ is a set of $n+1$ points in $\mathbb{R}^n$, then the center of the circumsphere, $\boldsymbol{r}$, can be obtained from the following set of linear equations:

$$2(\boldsymbol{r} - \boldsymbol{r}_0)(\boldsymbol{r}_i - \boldsymbol{r}_0) = (\boldsymbol{r}_i - \boldsymbol{r}_0)^2, \quad i = \overline{1, n}. \tag{4.1}$$

Orthogonal distance regression plane and line. Let $\{\boldsymbol{r}\}$ be a set of points in $\mathbb{R}^n$. The best plain (line) running through this set of points in the sense of minimizing the sum of the squared distances to the plain (line) corresponds to the minimum (maximum) eigenvalue of the matrix $\mathbf{A}$ given by

$$A_{ij} = \sum_{\boldsymbol{r}} (r_i - R_i)(r_j - R_j), \tag{4.2}$$

where $\boldsymbol{R}$ is the center of mass of the set which belongs to the regression plane (line). Namely, if α is the minimum eigenvalue of $\mathbf{A}$ and $\boldsymbol{e}$ is the corresponding normalized eigenvector, then $\boldsymbol{e}$ is the normal vector of the regression plain and the sum of the squared distances

$$\sum_{\boldsymbol{r}} (\boldsymbol{e}\delta\boldsymbol{r})^2 = \alpha, \tag{4.3}$$

where $\delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{R}$. If α is the maximum eigenvalue of $\mathbf{A}$ and $\boldsymbol{e}$ is the corresponding normalized eigenvector, then $\boldsymbol{e}$ is parallel to the regression line and the sum of the squared distances

$$\sum_{\boldsymbol{r}} (\delta\boldsymbol{r} - \boldsymbol{e}(\boldsymbol{e}\delta\boldsymbol{r}))^2 = \sum_{\boldsymbol{r}} \delta\boldsymbol{r}^2 - \alpha. \tag{4.4}$$

§5. Geometry on sphere

5.1. Basics

Each point on a sphere can be denoted as (θ, ϕ) where spherical coordinates $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ (or alternatively $\phi \in (-\pi, \pi)$). The extended coordinate system (x, y, z) can also be used, and the relation between the coordinates are as follows

$$x = \sin\theta\cos\phi,\ y = \sin\theta\sin\phi,\ z = \cos\theta, \qquad x^2 + y^2 + z^2 = 1.$$

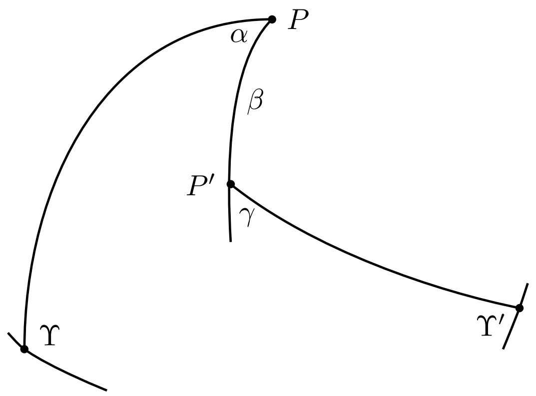

Figure 2: Coordinates transformation

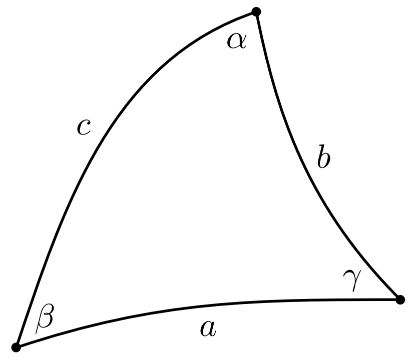

Figure 3: Spherical triangle

We also denote $P = (0, 0)$ to be the pole and $\Upsilon = (\pi/2, 0)$ to be the vernal equinox. Any directed path is assumed to be counterclockwise with respect to the selected point or the pole (right hand rule), any angle is counted counterclockwise from the direction to the pole.

Coordinates transformation to the pole P' in Euler parameters (α, β, γ) (see Fig. 2) are given by the formulas:

$$\cos\theta' = \cos\theta\cos\beta + \sin\theta\sin\beta\cos(\phi-\alpha), \qquad \tan(\phi'+\gamma) = \frac{\sin\theta\sin(\phi-\alpha)}{\sin\theta\cos\beta\cos(\phi-\alpha) - \cos\theta\sin\beta}, \tag{5.1}$$

$$\cos\theta = \cos\theta'\cos\beta - \sin\theta'\sin\beta\cos(\phi'+\gamma), \quad \tan(\phi-\alpha) = \frac{\sin\theta'\sin(\phi'+\gamma)}{\sin\theta'\cos\beta\cos(\phi'+\gamma) + \cos\theta'\sin\beta}. \tag{5.2}$$

One of the basic functions is the angle separation between two points:

$$\cos s_{12} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) \equiv x_1x_2 + y_1y_2 + z_1z_2. \tag{5.3}$$

5.2. Arcs and arc segments

A directed arc segment is the shortest (or complemented to the shortest) directed path between two points. A directed arc can be specified by its pole (θ_0, ϕ_0) perpendicular to each point of the arc. Hence the parametric form of the arc is $\cot\theta = -\tan\theta_0\cos(\phi-\phi_0)$. The distance from given point to an arc complements $\pi/2$ with the distance to the arc's pole.

The directed arc segment connecting given points can be specified by its pole (θ_0, ϕ_0) via the formulas:

$$\cos\theta_0 = \sin\theta_1\sin\theta_2\frac{\sin(\phi_2-\phi_1)}{\sin s_{12}}, \quad \tan\phi_0 = -\frac{\cot\theta_2\cos\phi_1 - \cot\theta_1\cos\phi_2}{\cot\theta_2\sin\phi_1 - \cot\theta_1\sin\phi_2}. \tag{5.4}$$

The parametric form is given by

$$\cot\theta = \frac{\cot\theta_2\sin(\phi-\phi_1) + \cot\theta_1\sin(\phi_2-\phi)}{\sin(\phi_2-\phi_1)}, \ \phi \in (\phi_1, \phi_2). \tag{5.5}$$

Note that (5.4) gives also the intersection point of two arcs.

The directed arc perpendicular to the segment connecting given points and dividing this segment in half has the pole with

$$\cos\theta_0 = \frac{\cos\theta_1 - \cos\theta_2}{2\sin(s_{12}/2)}, \quad \tan\phi_0 = \frac{\sin\theta_1\sin\phi_1 - \sin\theta_2\sin\phi_2}{\cos\theta_1\cos\phi_1 - \sin\theta_2\cos\phi_2}, \tag{5.6}$$

and this pole is closer to point 1. The middle of the segment connecting given points has coordinates

$$\cos\theta_0 = \frac{\cos\theta_1 + \cos\theta_2}{2\cos(s_{12}/2)}, \quad \tan\phi_0 = \frac{\sin\theta_1\sin\phi_1 + \sin\theta_2\sin\phi_2}{\sin\theta_1\cos\phi_1 + \sin\theta_2\cos\phi_2}. \tag{5.7}$$

5.3. Triangles and spherical trigonometry

A directed triangle is specified by an ordered triplet of its vertices. Denote α, β, γ to be the angles at the vertices 1, 2, 3 respectively and a, b, c to be the arc segments opposite to vertices 1, 2, 3 (see Fig. 3). Two basic theorems of spherical trigonometry are the cosine theorem:

$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha, \tag{5.8}$$

and the sine theorem:

$$\frac{\sin\alpha}{\sin a} = \frac{\sin\beta}{\sin b} = \frac{\sin\gamma}{\sin c} = \frac{(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})}{\sin a \sin b \sin c}, \tag{5.9}$$

where $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ is the triple scalar product equal to six volumes of the tetrahedron on these vectors.

There are also different derived formulas. The cosine theorem for angles:

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a, \tag{5.10}$$

which shows that in spherical trigonometry angles and segments are equivalent. The tangent theorem:

$$\frac{\tan\frac{1}{2}(\beta-\gamma)}{\tan\frac{1}{2}(\beta+\gamma)} = \frac{\tan\frac{1}{2}(b-c)}{\tan\frac{1}{2}(b+c)}. \tag{5.11}$$

Two useful identities:

$$\sin a\cos\gamma = \sin b\cos c - \cos b\sin c\cos\alpha \implies \sin\alpha\cot\gamma = \sin b\cot c - \cos b\cos\alpha.$$

Other formulas can be found at Wolfram.

For rectangular triangle, $\gamma = \pi/2$, one has Pythagorean theorem to be $\cos c = \cos a\cos b$ and basic trigonometric identities to be

$$\sin\alpha = \frac{\sin a}{\sin c}, \quad \cos\alpha = \frac{\tan b}{\tan c}, \quad \tan\alpha = \frac{\tan a}{\sin b}.$$

The triangle area or the so called spherical excess is given by

$$S = \alpha + \beta + \gamma - \pi. \tag{5.12}$$

Denote $s = S/2$ to be the semi-area and $p = (a+b+c)/2$ to be the semi-perimeter of the triangle. Note that

$$\tan\frac{s}{2} = \sqrt{\tan\frac{p}{2}\tan\frac{p-a}{2}\tan\frac{p-b}{2}\tan\frac{p-c}{2}}.$$

The circumcircle's radius R and the incircle's radius r are given by

$$\cot R = \sqrt{\frac{\sin(\alpha-s)\sin(\beta-s)\sin(\gamma-s)}{\sin s}}, \tag{5.13}$$

$$\tan r = \sqrt{\frac{\sin(p-a)\sin(p-b)\sin(p-c)}{\sin p}}. \tag{5.14}$$

Note that the angle between the side a and radius R is equal to $(\beta+\gamma-\alpha)/2$. The central point of the triangle, that is the circumcircle's center, can be found via the formulas:

$$\begin{aligned} \cot\theta_0 &= \frac{\sin\theta_1\sin\theta_2\sin(\phi_2-\phi_1) + \{123\to231\} + \{231\to312\}}{\sqrt{2}\sqrt{(1-\cos s_{12})(\cos\theta_1-\cos\theta_3)(\cos\theta_2-\cos\theta_3) + \{123\to231\} + \{231\to312\}}}, \\ \tan\phi_0 &= -\frac{\sin\theta_1(\cos\theta_2-\cos\theta_3)\cos\phi_1 + \{123\to231\} + \{231\to312\}}{\sin\theta_1(\cos\theta_2-\cos\theta_3)\sin\phi_1 + \{123\to231\} + \{231\to312\}}. \end{aligned} \tag{5.15}$$

§6. Vector calculus

Differential and Taylor's formula:

$$\mathrm{d}f = (\,\mathrm{d}\boldsymbol{r}\ \boldsymbol{\nabla})\, f, \qquad f\,(\boldsymbol{r}+\Delta\boldsymbol{r}) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(\Delta\boldsymbol{r}\ \boldsymbol{\nabla})^{n}\, f(\boldsymbol{r}).$$

Integral theorems. Surface or Ostrogradsky theorem:

$$\oint_{\partial V} \mathrm{d}\boldsymbol{S}\ f = \int_V \mathrm{d}V\ \boldsymbol{\nabla} f, \qquad \oint_{\partial V} \mathrm{d}\boldsymbol{S}\ \boldsymbol{A} = \int_V \mathrm{d}V\ \boldsymbol{\nabla}\boldsymbol{A}, \qquad \oint_{\partial V} \mathrm{d}\boldsymbol{S}\times\boldsymbol{A} = \int_V \mathrm{d}V\ \boldsymbol{\nabla}\times\boldsymbol{A}.$$

Contour or Stocks theorem:

$$\oint_{\partial S} \mathrm{d}\boldsymbol{r}\ f = \int_S \mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla} f\ , \qquad \oint_{\partial S} \mathrm{d}\boldsymbol{r}\ \boldsymbol{A} = \int_S \mathrm{d}\boldsymbol{S}\,(\boldsymbol{\nabla}\times\boldsymbol{A})\,, \qquad \oint_{\partial S} \mathrm{d}\boldsymbol{r}\times\boldsymbol{A} = \int_S (\,\mathrm{d}\boldsymbol{S}\times\boldsymbol{\nabla})\times\boldsymbol{A}.$$

Integration by parts:

$$\int_V (\boldsymbol{A}\boldsymbol{\nabla})\,\boldsymbol{B}\,\mathrm{d}V = \oint_{\partial V} \boldsymbol{B}\,(\boldsymbol{A}\,\mathrm{d}\boldsymbol{S}) - \int_V \boldsymbol{B}\,(\boldsymbol{\nabla}\boldsymbol{A})\ \mathrm{d}V.$$

Some identities with nabla operator. Differentiation of a product:

$$\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{A}\boldsymbol{B}) &= (\boldsymbol{A}\boldsymbol{\nabla})\,\boldsymbol{B} + (\boldsymbol{B}\boldsymbol{\nabla})\,\boldsymbol{A} + \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\,,\\
\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) &= (\boldsymbol{B}\boldsymbol{\nabla})\,\boldsymbol{A} - (\boldsymbol{A}\boldsymbol{\nabla})\,\boldsymbol{B} + \boldsymbol{A}\,(\boldsymbol{\nabla}\boldsymbol{B}) - \boldsymbol{B}\,(\boldsymbol{\nabla}\boldsymbol{A})\,,\\
(\boldsymbol{A}\boldsymbol{\nabla})\,\boldsymbol{A} &= \frac{1}{2}\boldsymbol{\nabla} A^2 - \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\,,\\
\boldsymbol{\nabla}(\boldsymbol{A}\times\boldsymbol{B}) &= \boldsymbol{B}\,(\boldsymbol{\nabla}\times\boldsymbol{A}) - \boldsymbol{A}\,(\boldsymbol{\nabla}\times\boldsymbol{B})
\end{aligned}$$

The second differentiation:

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} f) \equiv \Delta f, \qquad \boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0, \qquad \boldsymbol{\nabla}(\boldsymbol{\nabla}\times\boldsymbol{A}) = 0, \qquad \boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{A}) = \Delta\boldsymbol{A} + \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\,.$$

If f and $\boldsymbol{A}$ have finite discontinuity at surface $\psi(\boldsymbol{r}) = 0$ then singular parts of the derivatives are following:

$$\begin{aligned}
\boldsymbol{\nabla} f_{\text{sing}} &= (\boldsymbol{\nabla}\psi(\boldsymbol{r})\,[f_+ - f_-])\,\delta\,(\psi(\boldsymbol{r}))\,,\\
\boldsymbol{\nabla}\boldsymbol{A}_{\text{sing}} &= (\boldsymbol{\nabla}\psi(\boldsymbol{r})\,[\boldsymbol{A}_+ - \boldsymbol{A}_-])\,\delta\,(\psi(\boldsymbol{r}))\,,\\
\boldsymbol{\nabla}\times\boldsymbol{A}_{\text{sing}} &= (\boldsymbol{\nabla}\psi(\boldsymbol{r})\times[\boldsymbol{A}_+ - \boldsymbol{A}_-])\,\delta\,(\psi(\boldsymbol{r}))\,.
\end{aligned}$$

Some special cases. Composite function of argument $\boldsymbol{r}\phi(r)$:

$$\begin{aligned}
\boldsymbol{\nabla} f\,(\boldsymbol{r}\phi(r)) &= \phi\boldsymbol{\nabla} f + \phi'\frac{\boldsymbol{r}}{r}\,(\boldsymbol{r}\boldsymbol{\nabla})\,f,\\
\boldsymbol{\nabla}\boldsymbol{A}\,(\boldsymbol{r}\phi(r)) &= \phi\boldsymbol{\nabla}\boldsymbol{A} + \phi'\frac{\boldsymbol{r}}{r}\,((\boldsymbol{r}\boldsymbol{\nabla})\,\boldsymbol{A})\,,\\
\boldsymbol{\nabla}\times\boldsymbol{A}\,(\boldsymbol{r}\phi(r)) &= \phi\,(\boldsymbol{\nabla}\times\boldsymbol{A}) + \phi'\left[\frac{\boldsymbol{r}}{r}\times((\boldsymbol{r}\boldsymbol{\nabla})\,\boldsymbol{A})\right],
\end{aligned}$$

where $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla}\boldsymbol{A}$ means the differentiation with respect to the argument $\boldsymbol{r}\phi(r)$ as the whole. Scalar argument and modifications:

$$\boldsymbol{\nabla}\phi(r) = \phi'\frac{\boldsymbol{r}}{r}, \qquad \boldsymbol{\nabla}(\boldsymbol{r}\phi(r)) = 3\phi + \phi' r, \qquad \boldsymbol{\nabla}\times(\boldsymbol{r}\phi(r)) = 0, \qquad (\boldsymbol{A}\boldsymbol{\nabla})\,(\boldsymbol{r}\phi(r)) = \phi\boldsymbol{A} + \phi'\frac{\boldsymbol{r}}{r}\,(\boldsymbol{r}\boldsymbol{A})\,.$$

Other useful identities:

$$\boldsymbol{\nabla}\left(\frac{\boldsymbol{r}\boldsymbol{a}}{r^3}\right) = \boldsymbol{\nabla}\times\left(\frac{\boldsymbol{r}\times\boldsymbol{a}}{r^3}\right), \qquad \int_V \boldsymbol{r}\times(\boldsymbol{\nabla}\times\boldsymbol{A})\ \mathrm{d}V = 2\int_V \boldsymbol{A}\,\mathrm{d}V + \oint_{\partial V} \boldsymbol{r}\times(\,\mathrm{d}\boldsymbol{S}\times\boldsymbol{A})\,.$$

Three simple classes of vector fields:

1. Let $\boldsymbol{A} = \boldsymbol{r} f$. Then $\boldsymbol{\nabla}\times\boldsymbol{A} = -\boldsymbol{r}\times\boldsymbol{\nabla} f$ and is perpendicular to $\boldsymbol{A}$. In this case the curl is zero iff $f = f(r)$ that is the vector field $\boldsymbol{A}$ is radial.

2. Now let $\boldsymbol{A} = (\boldsymbol{e}_z\times\boldsymbol{r}) f$. Then $\boldsymbol{\nabla}\times\boldsymbol{A} = (\boldsymbol{r}\boldsymbol{\nabla} f + 2)\,\boldsymbol{e}_z - (\boldsymbol{e}_z\boldsymbol{\nabla} f)\,\boldsymbol{r}$ and is perpendicular to $\boldsymbol{A}$. In this case the curl is directed along z-axis iff $f = f(x, y)$ that is the vector field $\boldsymbol{A}$ is planar, e.g. circular. For the divergence $\boldsymbol{\nabla}\boldsymbol{A} = (\boldsymbol{e}_z\times\boldsymbol{r})\boldsymbol{\nabla} f$ and is zero iff $f = f(r, z)$.

3. Finally let $\boldsymbol{A} = \boldsymbol{A}(z)$. Then $\boldsymbol{\nabla}\times\boldsymbol{A}$ is perpendicular to z-axis. It is parallel to $\boldsymbol{A}$ iff $A_z = 0$ and $|\boldsymbol{A}|$ is constant. In this case the vector field $\boldsymbol{A}$ is helicoidal. For the divergence $\boldsymbol{\nabla}\boldsymbol{A}$ is zero iff A_z is constant.