# Handbook on basic analysis

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## §1. Constants

$\pi \approx 3.1416, \quad \mathrm{e} \approx 2.7183, \quad \ln 10 \approx 2.3026, \quad \lg 2 \approx 0.30103.$
Euler's constant $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^n k^{-1} - \ln n\right) \approx 0.5772.$

## §2. Sums and products

### 2.1. Series expansion

Taylor's formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(y)(x-y)^n \,\mathrm{d}y = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1},\ 0 < \xi < x.$$

Multidimensional Taylor's formula:

$$f(x_1,\dots,x_d) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{i=1}^{d} x_i \partial_i\right)^n f = \sum_{n_1,\dots,n_d \geqslant 0} f^{(n_1,\dots,n_d)} \frac{x_1^{n_1}\dots x_d^{n_d}}{n_1!\dots n_d!}.$$

Series expansion of some elementary functions:

$$(1+x)^\mu = \sum_{n=0}^{\infty} \binom{\mu}{n} x^n = 1 + \mu x + \frac{\mu(\mu-1)}{2} x^2 + \ldots, \qquad \frac{1}{(1-x)^{m+1}} = \sum_{n=0}^{\infty} \binom{m+n}{m} x^n.$$

## 2.2. Finite sums and products

Hypergeometric sums:

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}, \qquad \sum_{k=0}^{n} k x^k = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2},$$

$$\sum_{k=0}^{\infty} q^k (\cos kx + \mathrm{i}\sin kx) = \frac{1-q\cos x + \mathrm{i}q\sin x}{1-2q\cos x+q^2}, \ |q|<1.$$

"Number of points" sums:

$$\sum_{1\leqslant k_1\leqslant k_2\leqslant\ldots\leqslant k_m\leqslant n} 1 = \binom{n+m-1}{m}, \qquad \sum_{1\leqslant k_1<k_2<\ldots<k_m\leqslant n} 1 = \binom{n}{m}.$$

Pascal's triangle for multinomial coefficients:

$$\sum_{k=0}^{m} \frac{(n-1)!}{n_1!\ldots n_{k-1}!(n_k-1)!n_{k+1}!\ldots n_m!} = \frac{n!}{n_1!\ldots n_m!}, \qquad n = n_1+\ldots+n_m.$$

## 2.3. Infinite sums and products

Polynomial products:

$$\prod_{n=0}^{\infty}\left(1+\frac{y^2}{(x+n)^2}\right) = \left|\frac{\Gamma(x)}{\Gamma(x+\mathrm{i}y)}\right|^2, \qquad \prod_{n\in\mathbb{Z}} \frac{(x_1+n)^2+y_1^2}{(x_2+n)^2+y_2^2} = \frac{\sin^2\pi x_1+\sinh^2\pi y_1}{\sin^2\pi x_2+\sinh^2\pi y_2}.$$

Products reducible to theta function:

$$\prod_{n=1}^{\infty}\left(1-q^{2n}\right) = \sqrt[3]{\frac{\theta_1'}{2\sqrt[4]{q}}}, \quad \prod_{n=0}^{\infty}\left(1-q^{2n+1}\right) = \sqrt[6]{\frac{2\sqrt[4]{q}\theta_4^3}{\theta_1'}}, \quad \prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1-2q^{2n}\cos 2x+q^{4n}\right) = \frac{\theta_1(x)}{2\sqrt[4]{q}\sin x},$$

some other products can be obtained via the identity $1+q^n = \frac{1-q^{2n}}{1-q^n}$.

Miscellaneous products:

$$\prod_{n=0}^{\infty}\left(1+x^{2^n}\right) = \frac{1}{1-x}.$$

Complex variables method:

$$\sum_{n\in\mathbb{Z}} (-1)^n f(n) = \frac{1}{2\mathrm{i}} \int_{\mathcal{C}} \frac{f(z)\,\mathrm{d}z}{\sin\pi z},$$

where the contour $\mathcal{C}$ encircles the whole real line and $f$ has no singular points inside $\mathcal{C}$.

## 2.4. Continued fractions

Continued fraction is defined by

$$\mathrm{K}_{k=1}^{n}\left(a_k/b_k\right) = \frac{a_1}{b_1+}\,\frac{a_1}{b_2+}\ldots\frac{a_n}{b_n} = \cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots\cfrac{a_n}{b_n}}} \equiv \frac{A_n}{B_n},$$

where $A_n$ and $B_n$ are solutions of the three-term linear recurrence

$$X_n = b_n X_{n-1} + a_n X_{n-2}$$

with the initial conditions $A_0 = 0$, $A_1 = a_1$ and $B_0 = 1$, $B_1 = b_1$.

# §3. Integrals

## 3.1. Indefinite integrals

Trigonometric:

$$\int R(\sin x, \cos x)\,\mathrm{d}x = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{\mathrm{d}t}{1+t^2}, \quad t = \tan\frac{x}{2}.$$

Exponential and trigonometric:

$$\int x\begin{pmatrix}\sin ax\\ \cos ax\end{pmatrix}\mathrm{d}x = \frac{1}{a^2}\begin{pmatrix}\sin ax\\ \cos ax\end{pmatrix} \mp \frac{x}{a}\begin{pmatrix}\cos ax\\ \sin ax\end{pmatrix},$$

$$\int \mathrm{e}^{ax}\begin{pmatrix}\sin bx\\ \cos bx\end{pmatrix}\mathrm{d}x = \frac{\mathrm{e}^{ax}}{a^2+b^2}\left[a\begin{pmatrix}\sin bx\\ \cos bx\end{pmatrix} \mp b\begin{pmatrix}\cos bx\\ \sin bx\end{pmatrix}\right],$$

$$\int x^n \mathrm{e}^{ax}\,\mathrm{d}x = \mathrm{e}^{ax}\left(\frac{x^n}{a} - \frac{nx^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} - \dots\right).$$

Miscellaneous integrals:

$$\binom{n}{m} m \int_0^x y^{m-1}(1-y)^{n-m}\,\mathrm{d}y = \sum_{k=m}^{n}\binom{n}{k} x^k (1-x)^{n-k}.$$

## 3.2. Definite integrals

Exponential:

$$\int_{-\infty}^{x} \mathrm{e}^{-ax^2+bx}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}\left(1+\operatorname{erf}\left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)\right) \underset{x\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}},$$

$$\int_0^\infty x^{s\mu-1}\mathrm{e}^{-ax^s}\,\mathrm{d}x = \frac{\Gamma(\mu)}{sa^\mu}, \qquad \int_0^\infty x^{\mu-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x = \Gamma(\mu)\psi(\mu),$$

$$\int_0^\infty \mathrm{e}^{-ax^2+bx}\frac{\mathrm{d}x}{\sqrt{x}} = \pi\sqrt{\frac{|b|}{8a}}\,\mathrm{e}^{\frac{b^2}{8a}}\left[I_{-1/4}\left(\frac{b^2}{8a}\right) + \operatorname{sgn} q\, I_{1/4}\left(\frac{b^2}{8a}\right)\right].$$

Trigonometric:

$$\int_0^{\pi/2}\sin^\mu x\cos^\nu x\,\mathrm{d}x = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{\mu+\nu}{2}+1\right)}, \qquad \int_0^\pi \frac{\mathrm{d}x}{\sqrt{1-2k\cos x+k^2}} = 2\mathrm{K}(k),\ |k|<1,$$

$$\int_0^\pi \frac{\cos nx\,\mathrm{d}x}{(1-2a\cos x+a^2)^{m+1}} = \frac{\pi a^n}{(1-a^2)^{m+1}}\sum_{k=0}^{m}\binom{m+n}{k+n}\binom{m+k}{k}\left(\frac{a^2}{1-a^2}\right)^k,\ |a|<1.$$

Fourier:

$$\tanh a = 2\int_0^\infty \frac{\sin 2ax}{\sinh \pi x}\,\mathrm{d}x,$$

$$\coth a = \frac{1}{a} + 2\int_0^\infty \sin 2ax(\coth\pi x - 1)\,\mathrm{d}x,$$

$$\ln\sinh a = \ln a + \int_0^\infty (1-\cos 2ax)(\coth\pi x - 1)\frac{\mathrm{d}x}{x}.$$

## 3.3. Multiple integrals

The volume of a ball and the area of a sphere in $\mathbb{R}^n$ are given by

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}, \qquad S_n = nV_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

Miscellaneous integrals:

$$\int_{\mathbb{R}^3} r^{\mu-2} \exp\left(-ar + \mathrm{i}\boldsymbol{k}\boldsymbol{r}\right) \mathrm{d}V = \frac{4\pi\Gamma(\mu)\sin\mu\phi}{k\,(a^2+k^2)^{\mu/2}}, \ \tan\phi = \frac{k}{a},$$

$$\int_{\mathbb{R}^n} \exp\left(-\alpha r^2 + \mathrm{i}\boldsymbol{k}\boldsymbol{r}\right) \mathrm{d}V = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} \exp\left(-\frac{k^2}{4\alpha}\right).$$

### Integrals on quadratic form

Let consider integrals of the kind

$$I(x'') = \sqrt{\det A} \int_{\mathbb{R}^k} f\left(\sqrt{(x, Ax)};\ x\right) \mathrm{d}x',$$

where $x \in \mathbb{R}^n$ and $(x, Ax) = \sum_{i,j=1}^n A_{ij} x_i x_j$ with a symmetric positively defined matrix $A$. The $\mathbb{R}^k$ subspace is denoted by the prime and the $\mathbb{R}^{n-k}$ subspace is denoted by the double prime, so that $x = x' + x''$. Let

$$A = \begin{pmatrix} A' & M \\ M^\top & A'' \end{pmatrix}, \qquad A^{-1} = C = \begin{pmatrix} C' & N^\top \\ N & C'' \end{pmatrix},$$

so that $(x, Ax) = (x', A'x') + (x'', A''x'') + 2(x', Mx'')$. The integral $I$ can be simplified by the diagonalization of $\mathbb{R}^k$ subspace produced by the substitution $x' = Sy - C'Mx''$, where $S^{-2} = A'$. The result is

$$I(x'') = \sqrt{\det \tilde{A}} \int_{\mathbb{R}^k} f\left(\sqrt{(y, y) + (x'', \tilde{A}x'')};\ Sy - C'Mx'' + x''\right) \mathrm{d}y,$$

where $\tilde{A} = (C'')^{-1}$. In particular,

$$\sqrt{\det A} \int_{\mathbb{R}^n} f\left(\sqrt{(x, Ax)}, (a, x)\right) \mathrm{d}x = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \int_0^\infty \int_0^\pi f(r, |Sa| r\cos\theta) r^{n-1} \sin^{n-2}\theta \,\mathrm{d}r\,\mathrm{d}\theta,$$

$$\sqrt{\det A} \int_{\mathbb{R}^n} f\left(\sqrt{(x, Ax)}\right) \mathrm{d}x = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty f(r) r^{n-1}\,\mathrm{d}r,$$

$$\sqrt{\det A} \int_{\mathbb{R}^n} x_i x_j f\left(\sqrt{(x, Ax)}\right) \mathrm{d}x = C_{ij} \frac{2\pi^{n/2}}{n\Gamma(n/2)} \int_0^\infty f(r) r^{n+1}\,\mathrm{d}r.$$

### Gauss integrals

Let us consider the following $n$-dimensional integral

$$I_k(a) = \frac{1}{\sqrt{\det(2\pi C)}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}(x,Ax)+(a,x)} \sum_{i_1 \ldots i_k} B_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k}\,\mathrm{d}x,$$

where the notations are the same as above. Substitution $x = y + Ca$ reduces this integral to

$$I_k(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)} \frac{1}{\sqrt{\det(2\pi C)}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}(y,Ay)} \sum_{i_1 \ldots i_k} B_{i_1 \ldots i_k} \left(y_{i_1} + (Ca)_{i_1}\right) \ldots \left(y_{i_k} + (Ca)_{i_k}\right) \mathrm{d}y.$$

Integral $I_k(0)$ can be calculated explicitly: it will zero for odd $k$ and

$$I_{2m}(0) = (2m-1)!! \sum_{i_1 \ldots i_{2m}} B_{[i_1 i_2 \ldots i_{2m}]} C_{i_1 i_2} C_{i_3 i_4} \ldots C_{i_{2m-1} i_{2m}} \equiv (2m-1)!! \sum_{i_1 \ldots i_{2m}} B_{i_1 i_2 \ldots i_{2m}} C_{[i_1 i_2} C_{i_3 i_4} \ldots C_{i_{2m-1} i_{2m}]},$$

where $[i_1 i_2 \ldots i_{2m}]$ means the symmetrization over the all indexes, i.e.

$$B_{[i_1 i_2 \ldots i_{2m}]} \equiv \frac{1}{(2m)!} \sum_{j_1 j_2 \ldots j_{2m} = \mathrm{perm}(i_1 i_2 \ldots i_{2m})} B_{j_1 j_2 \ldots j_{2m}}.$$

For example,

$$\frac{1}{\sqrt{\det(2\pi C)}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}(x,Ax)} x_i^{2m}\,\mathrm{d}V = (2m-1)!! C_{ii}^m.$$

For small $k$ we have

$$I_0(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)}, \qquad I_1(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)} (B, Ca), \qquad I_2(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)} \left[\mathrm{tr}(BC) + (a, CBCa)\right].$$

## §4. Asymptotic methods

The directed equality $f(x) = O(\varphi(x)),\ x \to 0$ means $\exists a, c\ \forall x : |x| < a\ |f(x)| \leqslant c|\varphi(x)|$. If no limiting point is specified then the equivalence is considered as uniform. The directed equality $f(x) = o(\varphi(x)),\ x \to 0$ means $\lim_{x\to 0} f(x)/\varphi(x) = 0$. The equality $f(x) \sim \varphi(x),\ x \to 0$ means $\lim_{x\to 0} f(x)/\varphi(x) = 1$. The asymptotic expansion $f(x) \sim \sum_{k=1}^{\infty} c_k\varphi_k(x),\ x \to 0$ means $f(x) - \sum_{k=1}^{n} c_k\varphi_k(x) = o(\varphi_n(x)),\ x \to 0$. These relations are algebraically transitive and admit integration, differentiation is generally not allowed.

### 4.1. Summation

Sums with smoothly varying terms are evaluated by Euler–Maclaurin formula:

$$\sum_{k=0}^{n} f(hk) = \frac{1}{h}\int_0^{hn} f(x)\,\mathrm{d}x + \frac{f(0)+f(hn)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(0) - f^{(2k-1)}(hn)\right] h^{2k-1} - h^{2m}\int_0^{n} \frac{\tilde{B}_{2m}(y)}{(2m)!} f^{(2m)}(hy)\,\mathrm{d}y,$$

where

$$\frac{\tilde{B}_{2m}(y)}{(2m)!} = 2(-1)^{m+1}\sum_{k=1}^{\infty}\frac{\cos 2\pi k y}{(2\pi k)^{2m}}.$$

For sums with oscillating terms there are following methods. In case of alternating terms the sum can be converted to the one with positive terms by abelian transformation:

$$\sum_{k=0}^{n} a_k b_k = A_n b_n + \sum_{k=0}^{n-1} A_k(b_k - b_{k+1}), \qquad A_n = \sum_{k=0}^{n} a_k.$$

In other cases one can use Poisson's summation formula:

$$\sum_{k=-\infty}^{\infty} f(k+a) = \lim_{n\to\infty}\sum_{k=-n}^{n} \mathrm{e}^{2\pi \mathrm{i} k a}\int_{-\infty}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} k x} f(x)\,\mathrm{d}x,$$

providing that the sums $\sum_k f(k+x)$ and $\sum_k f'(k+x)$ converge uniformly for $0 \leqslant x \leqslant 1$.

See also abelian and tauberian theorems in Section 6.3.

### 4.2. Integration

See abelian and tauberian theorems in Section 5.2.

### 4.3. Saddle-point methods

Here we consider asymptotics of integral $\int \mathrm{e}^{\lambda S(z)} f(z)\,\mathrm{d}z$ as $\lambda \to +\infty$.

Let $x_0$ be a nondegenerate global maximum of $S$ and let $x_0$ be an inner point of the integration interval then

$$\int \mathrm{e}^{\lambda S(x)} f(x)\,\mathrm{d}x \sim \mathrm{e}^{\lambda S(x_0)}\sqrt{\frac{2\pi}{-\lambda S''(x_0)}}\left[\sum_{k=0}^{3n} c_{2k}(\lambda)\left(\frac{1}{2}\right)_k \left(\frac{2}{-\lambda S''(x_0)}\right)^k + o\left(\lambda^{-n}\right)\right], \tag{4.1}$$

where $c_m$ are defined via the following generating function

$$f(x)\exp\lambda\left[S(x) - S(x_0) - \frac{1}{2}S''(x_0)(x-x_0)^2\right] = \sum_{m=0}^{\infty} c_m(\lambda)(x-x_0)^m.$$

If $x_0$ coincides with the boundary point of the integration interval then one must divide the right-hand side of (4.1) by 2. The first two terms in (4.1) can be written explicitly:

$$\int \mathrm{e}^{\lambda S(x)} f(x)\,\mathrm{d}x \sim \mathrm{e}^{\lambda S(x_0)}\sqrt{\frac{2\pi}{-\lambda S''(x_0)}}\left[f + \frac{1}{\lambda}\left(-\frac{f''}{2S''} + \frac{fS''''}{8S''^2} + \frac{f'S'''}{2S''^2} - \frac{5fS'''^2}{24S''^3}\right) + \dots\right]_{x=x_0}.$$

In a multidimensional case

$$\int \mathrm{e}^{\lambda S(x)} f(x)\,\mathrm{d}x \sim \frac{\mathrm{e}^{\lambda S(x_0)}}{\sqrt{\det \frac{\lambda Q}{2\pi}}} \sum_{k=0}^{\infty} \frac{1}{k! 2^k \lambda^k} \left(Q^{-1}_{ij} \partial^i \partial^j\right)^k \left\{ f(x) \exp \lambda \left[ S(x) - S(x_0) - \frac{1}{2} Q_{ij} (x^i - x^i_0)(x^j - x^j_0) \right] \right\}\Big|_{x=x_0}.$$

where $Q_{ij} = -\left.\frac{\partial^2 S}{\partial x^i \partial x^j}\right|_{x_0}$ and the summation over the dummy indexes is used. The leading term in the above series is $f(x_0)$.

If $x_0$ is a degenerate maximum such that $S(x) - S(x_0) \sim (x - x_0)^{2m}$ then the leading term is

$$\int \mathrm{e}^{\lambda S(x)} f(x)\,\mathrm{d}x \sim \frac{1}{m} \Gamma\left(\frac{1}{2m}\right) \mathrm{e}^{\lambda S(x_0)} \sqrt[2m]{-\frac{(2m)!}{\lambda S^{(2m)}(x_0)}} f(x_0).$$

If the global maximum of $S$ is reached at the left boundary of the integration interval $a$ and $S'(a) \neq 0$ then

$$\int_a \mathrm{e}^{\lambda S(x)} f(x)\,\mathrm{d}x \sim \mathrm{e}^{\lambda S(a)} \sum_{k=0}^{\infty} \lambda^{-k-1} \left(-\frac{\mathrm{d}}{S'(x)\,\mathrm{d}x}\right)^k \left(-\frac{f(x)}{S'(x)}\right)\Big|_{x=a} = -\frac{\mathrm{e}^{\lambda S(a)}}{\lambda} \frac{f(a)}{S'(a)} + \dots$$

Some examples of oscillating integrals:

$$\int_0^{\pi-\varepsilon} \mathrm{e}^{\mathrm{i}\lambda \cos\phi} f(\phi)\,\mathrm{d}\phi \sim \sqrt{\frac{\pi}{2\lambda}} \mathrm{e}^{\mathrm{i}(\lambda - \pi/4)} f(0).$$

See also [2].

# §5. Integral transforms

## 5.1. Fourier transform

Fourier transform is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \ \ k \in \mathbb{C},$$

so that $\nabla \to \mathrm{i}k$ as in quantum mechanics. Its inverse is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \ \ x \in \mathbb{R}.$$

In mathematical literature the symmetric form is used, and the second integral is considered as principal value. Besides $(x,k)$ pair, in physical literature $(t,\omega)$ pair is used, but with opposite sign:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t,$$

to perform Fourier transform by plane waves $\mathrm{e}^{\mathrm{i}(kx - \omega t)}$.

Some identities:

$$\int_{-\infty}^{\infty} \overline{f(x)} g(x)\,\mathrm{d}x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\hat{f}(k)} \hat{g}(k)\,\mathrm{d}k,$$

$$\int_{-\infty}^{\infty} f(x-y) g(y)\,\mathrm{d}y \to \hat{f}(k) \hat{g}(k).$$

Transformation table ($n$ is the dimension):

$$\delta(x) \to 1,$$
$$\exp\left(-\alpha r^2\right) \to \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} \exp\left(-\frac{k^2}{4\alpha}\right),$$
$$\mathrm{e}^{-\alpha r} \to \frac{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) \alpha}{\left(\alpha^2+k^2\right)^{\frac{n+1}{2}}},$$
$$\frac{r^{m-\frac{n}{2}} K_{m-\frac{n}{2}}(\alpha r)}{2^{m+\frac{n}{2}-1} \pi^{\frac{n}{2}} \Gamma(m) \alpha^{m-\frac{n}{2}}} \to \frac{1}{(\alpha^2+k^2)^m}$$

$n$-dimensional Fourier transform of axially symmetric function $f(\boldsymbol{x}) = f(r,\theta)$ can be calculated by formulas

$$\hat{f}(k,\beta) = S_{n-2} \int_0^\infty r^{n-1}\,\mathrm{d}r \int_0^\pi f(r,\theta) \mathrm{e}^{-\mathrm{i}kr\cos\theta} \sin^{n-2}\theta\,\mathrm{d}\theta,$$
$$f(r,\theta) = \frac{S_{n-2}}{(2\pi)^n} \int_0^\infty k^{n-1}\,\mathrm{d}k \int_0^\pi \hat{f}(k,\beta) \mathrm{e}^{\mathrm{i}kr\cos\beta} \sin^{n-2}\beta\,\mathrm{d}\beta.$$

In the case of spherical symmetry

$$\hat{f}(k) = \frac{(2\pi)^{n/2}}{k^{\frac{n}{2}-1}} \int_0^\infty f(r) J_{\frac{n}{2}-1}(kr) r^{n/2}\,\mathrm{d}r, \qquad f(r) = \frac{1}{(2\pi)^{n/2} r^{\frac{n}{2}-1}} \int_0^\infty \hat{f}(k) J_{\frac{n}{2}-1}(kr) k^{n/2}\,\mathrm{d}k.$$

### 5.2. Laplace transform

Laplace transform is defined by

$$\tilde{f}(s) = \int_0^\infty f(t) \mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}F(t),\ s \in \mathbb{C},$$

where the second expression is a more general form with monotonic non-decreasing $F$ (if $F$ is differentiable then $f \equiv F'$). The inverse transformation is given by

$$f(t) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \tilde{f}(s) \mathrm{e}^{st}\,\mathrm{d}s,\ t \in \mathbb{R}_+.$$

The function $f(t)$ is assumed to be zero for $t < 0$, $\tilde{f}(s)$ must be analytic for $\Re s > \sigma$. If the only singularities of $\tilde{f}$ are branch cut $(-p, 0)$ and isolated singular points then one can use the following formula in the limit $r \to 0$:

$$f(t) = \sum_s \operatorname{res} \tilde{f}(s) \mathrm{e}^{st} + \frac{1}{2\pi\mathrm{i}} \int_r^{p-r} \left[\tilde{f}(-x-\mathrm{i}0) - \tilde{f}(-x+\mathrm{i}0)\right] \mathrm{e}^{-xt}\,\mathrm{d}x$$
$$+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{f}(r\mathrm{e}^{\mathrm{i}\phi}) \mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\phi}} r\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi + \frac{\mathrm{e}^{-pt}}{2\pi} \int_0^{2\pi} \tilde{f}(-p + r\mathrm{e}^{\mathrm{i}\phi}) \mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\phi}} r\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi.$$

Some properties:

- shift and dilatation:

$$f(t-a) \to \mathrm{e}^{-as} \tilde{f}(s),\ a \geqslant 0, \qquad \mathrm{e}^{at} f(t) \to \tilde{f}(s-a), \qquad f(at) \to \frac{1}{a} \tilde{f}\left(\frac{s}{a}\right),$$

- differentiation and integration:

$$f^{(n)}(t) \to s^n \tilde{f}(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{(n-1)}(0) = \frac{1}{s} \operatorname*{Regular}_{s\to\infty} \left[s^{n+1} \tilde{f}(s)\right],$$
$$t^n f(t) \to (-1)^n \tilde{f}^{(n)}(s), \qquad \int_0^t f(\tau)\,\mathrm{d}\tau \to \frac{1}{s} \tilde{f}(s), \qquad \frac{1}{t} f(t) \to \int_s^\infty \tilde{f}(\sigma)\,\mathrm{d}\sigma,$$

- convolution and product:

$$\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau \to \tilde{f}(s)\tilde{g}(s), \qquad f(t)g(t) \to \frac{1}{2\pi\mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} \tilde{f}(\sigma)g(s-\sigma)\,\mathrm{d}\sigma,$$

- special properties:

$$\int_0^\infty f(\tau)g(t,\tau)\,\mathrm{d}\tau \to \tilde{f}(q(s))\tilde{g}(s), \text{ where } g(t,\tau) \to \mathrm{e}^{-\tau q(s)}\tilde{g}(s),$$

$$\forall t \geqslant 0 \ f(t) \geqslant g(t) \implies \forall s \geqslant 0 \ \tilde{f}(s) \geqslant \tilde{g}(s).$$

Transformation table:

$$\mathrm{e}^{at} \to \frac{1}{(s-a)}, \qquad \cos at + \mathrm{i}\sin at \to \frac{s+\mathrm{i}a}{s^2+a^2}, \qquad t^{\mu-1} \to \frac{\Gamma(\mu)}{s^\mu}, \qquad t^{\mu-1}\left[\psi(\mu) - \ln t\right] \to \frac{\Gamma(\mu)}{s^\mu}\ln s,$$

$$\frac{1}{(t+a)^{\mu+1}} \to \mathrm{e}^{as}s^\mu\Gamma(-\mu, as), \ a > 0, \qquad \gamma(\mu, at)\mathrm{e}^{at} \to \frac{\Gamma(\mu)a^\mu}{s^\mu(s-a)}, \qquad J_\mu(at) \to \frac{\left(\sqrt{s^2+a^2}-s\right)^\mu}{a^\mu\sqrt{s^2+a^2}},$$

$$\mathrm{e}^{-a\sqrt{t}} \to \frac{1}{s}\left[1 - \frac{a\sqrt{\pi}}{2\sqrt{s}}\mathrm{e}^{\frac{a^2}{4s}}\left(1 - \operatorname{erf}\left(\frac{a}{2\sqrt{s}}\right)\right)\right], \qquad t^{\mu-1}\mathrm{e}^{-\frac{a}{t}} \to 2\left(\frac{a}{s}\right)^{\frac{\mu}{2}} K_\mu(2\sqrt{as}),$$

$$-\mathrm{Ei}\left(-\frac{t}{a}\right) \to \frac{\ln(1+as)}{s}, \qquad \mathrm{e}^t\mathrm{Li}_\mu\left(-\mathrm{e}^t\right) \to \frac{\pi}{(1+s)^\mu \sin\pi s}.$$

**Regular series**

Series expansion around $s = \infty$:

$$\tilde{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \ldots + \frac{f^{(n)}(0)}{s^{n+1}} + \ldots$$

Expansion in Laguerre polynomials can provide numerical inversion:

$$f(t) = t^a\sum_{n=0}^\infty c_n L_n^a(t), \text{ where } c_n = \frac{(-1)^n}{\Gamma(n+a+1)}\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\frac{1}{s^{a+1}}\tilde{f}\left(\frac{1}{s}\right)\right)\bigg|_{s=1}.$$

**Asymptotic expansion as $t \to \infty$**

By argument shift one can move any singularity to $s = 0$. If $s = 0$ is an isolated singular point then one can use the residue formula. In the case of non-isolated singularities we obtain asymptotic expansion as $t \to \infty$. In particular, let $s = 0$ be an algebraic branch point and thus

$$\tilde{f}(s) = s^\mu\sum_{n=0}^\infty c_n s^n$$

or linear combination of such generalized series. Then for $f(t)$ we obtain the following asymptotic expansion:

$$f(t) \sim -\frac{\sin\pi\mu}{\pi}\frac{1}{t^{\mu+1}}\sum_{n=0}^\infty \frac{(-1)^n c_n\Gamma(n+\mu+1)}{t^n}.$$

For simple logarithmic singularity one have similarly

$$\tilde{f}(s) = \ln s\sum_{n=0}^\infty c_n s^n \implies f(t) \sim -\frac{1}{t}\sum_{n=0}^\infty \frac{(-1)^n c_n\Gamma(n+1)}{t^n}.$$

For complex logarithmic singularities one can use the correspondence

$$s^n\ln^m s \leftarrow \frac{(-1)^m m!}{t^{n+1}}\sum_{k=0}^{\min(m-1,n)} S_{n+1}^{k+1}\left(\frac{t^z}{\Gamma(z)}\right)_{m-k},$$

where $S_n^k$ are Stirling's numbers of the first kind and $()_k$ is the $k$-th term in series expansion at $z = 0$, which can be obtained from

$$\frac{t^z}{\Gamma(z)} = \frac{\sin\pi z}{\pi}\exp\left[(\gamma + \ln t)\,z + \sum_{k=2}^\infty \frac{\zeta(k)}{k}z^k\right].$$

**Tauberian theorems**

Let call function $\varphi$ slow varying if $\forall\lambda>0\ \lim_{x\to\infty}\frac{\varphi(\lambda x)}{\varphi(x)}=1$ (logarithmic function and any function having a limit at infinity are examples). It can be shown [1, v.2, p.508] that

$$\tilde{f}(s)\sim s^{-\mu}\varphi(1/s),\ s\to+0 \quad\Longleftrightarrow\quad F(t)\sim\frac{t^{\mu}}{\Gamma(\mu+1)}\varphi(t),\quad t\to\infty,$$
$$\tilde{f}(s)\sim s^{-\mu}\varphi(s),\quad s\to\infty \quad\Longleftrightarrow\quad F(t)=\frac{t^{\mu}}{\Gamma(\mu+1)}\varphi(1/t),\ t\to+0.$$

If $F$ is differentiable then

$$s\tilde{f}(s)\sim s^{-\mu}\varphi(1/s),\ s\to+0 \Longleftrightarrow f(t)\sim\frac{t^{\mu}}{\Gamma(\mu+1)}\varphi(t),\ t\to\infty,$$

and $\lim_{s\to0}s\tilde{f}(s)=\langle f\rangle_{t\to\infty}$ if the limit exists.

**Laplace transforms of logarithmic functions**

One can show that

$$t^{\nu-1}\ln^m t\to\frac{\Gamma(\nu)}{s^{\nu}}P_m^{\nu}(\ln s)\qquad\text{and}\qquad\frac{1}{s^{\nu}}\ln^m s\leftarrow\frac{t^{\nu-1}}{\Gamma(\nu)}Q_m^{\nu}(\ln t),$$

where $m\in\mathbb{Z}_+$, $P_m^{\nu}$ and $Q_m^{\nu}$ are polynomials of the order $m$. Explicit formulas for these polynomials are

$$P_m^{\nu}(x)=\frac{\mathrm{e}^{\nu x}}{\Gamma(\nu)}\frac{\partial^m(\mathrm{e}^{-\nu x}\Gamma(\nu))}{\partial\nu^m},\qquad Q_m^{\nu}(x)=-\Gamma(\nu)\mathrm{e}^{-\nu x}\frac{\partial^m}{\partial\nu^m}\left(\frac{\mathrm{e}^{\nu x}}{\Gamma(\nu)}\right).$$

They can be found with the use of the recurrence relations

$$P_0^{\nu}(x)=+1,\qquad P_1^{\nu}(x)=\psi(\nu)-x,\qquad P_{m+1}^{\nu}(x)=\frac{\partial P_m^{\nu}(x)}{\partial\nu}+P_1^{\nu}(x)P_m^{\nu}(x),$$
$$Q_0^{\nu}(x)=-1,\qquad Q_1^{\nu}(x)=\psi(\nu)-x,\qquad Q_{m+1}^{\nu}(x)=\frac{\partial Q_m^{\nu}(x)}{\partial\nu}-Q_1^{\nu}(x)Q_m^{\nu}(x).$$

Note that the difference between $P$ and $Q$ is only in opposite signs at $\psi^{(2k+1)}(\nu),\ k\in\mathbb{Z}_+$. If $\nu\leqslant0$ then the transform is undefined. But we can easily define inverse Laplace transforms for these values of $\nu$ by the above formula, taking limit for non-positive integer $\nu$. Direct Laplace transform is still undefined for non-positive integer $\nu$ by the above formula, but it can be defined via the inverse transform in this case.

# §6. Discrete transforms

## 6.1. Fourier series

Fourier series is defined in context of the following transformation:

$$\hat{f}(k)=\sum_{n\in\mathbb{Z}}f_n\mathrm{e}^{\mathrm{i}kn},\ k\in(-\pi,\pi],$$
$$f_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(k)\mathrm{e}^{-\mathrm{i}kn}\,\mathrm{d}k,\ n\in\mathbb{Z}.$$

$\hat{f}(k)$ is regarded as $2\pi$-periodic so one can choose $k\in[0,2\pi)$ also.

Some properties:

- transform of unit function:

$$1\to2\pi\delta(k),\qquad\delta_{n0}\to1,$$

- argument shift and inversion:

$$f_{n+a} \to \hat{f}(k)\mathrm{e}^{-\mathrm{i}ka}, \qquad f_n\mathrm{e}^{\mathrm{i}pn} \to \hat{f}(k+p), \qquad f_{-n} \to \hat{f}(-k) \equiv \hat{f}(2\pi - k),$$

- multiplication on argument and differentiation:

$$nf_n \to -\mathrm{i}\frac{\mathrm{d}\hat{f}(k)}{\mathrm{d}k}, \qquad f_n - f_{n-1} \to \hat{f}(k)(1-\mathrm{e}^{\mathrm{i}k}),$$

- product and convolution:

$$f_n g_n \to \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(\kappa)\hat{g}(k-\kappa)\,\mathrm{d}\kappa, \qquad \sum_{m\in\mathbb{Z}} f_m g_{n-m} \to \hat{f}(k)\hat{g}(k),$$

- other identities:

$$\sum_{n\in\mathbb{Z}} f_n g_n = \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(k)\hat{g}(2\pi-k)\,\mathrm{d}k, \qquad \sum_{n\in\mathbb{Z}} f_n g_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(k)\hat{g}(k)\,\mathrm{d}k.$$

Transformation table:

$$\mathrm{e}^{-\alpha|n|} \to \frac{1-q^2}{1-2q\cos k + q^2}.$$

### 6.2. Discrete Fourier transform

Discrete Fourier transform is defined by

$$\hat{f}_k = \sum_{n=0}^{L-1} f_n \mathrm{e}^{\mathrm{i}kn},\ k = \frac{2\pi}{L}l,\ l = \overline{0, L-1,},$$
$$f_n = \frac{1}{L}\sum_k \hat{f}_k \mathrm{e}^{-\mathrm{i}kn},\ n = \overline{0, L-1,},$$

both $f_n$ and $\hat{f}_k$ are $L$-periodic. If $\hat{f}_k$ is even function then

$$f_n = \frac{1}{L}\hat{f}_0 + \frac{2}{L}\sum_{l=1}^{\left[\frac{L-1}{2}\right]} \hat{f}_l \cos\left(\frac{2\pi nl}{L}\right) + \mathcal{I}\left\{L \text{ is even}\right\}\frac{(-1)^n}{L}\hat{f}_{\frac{L}{2}}.$$

Some properties:

- transform of unit function:

$$1 \to L\delta_{k0}, \qquad \delta_{n0} \to 1,$$

- argument shift:

$$f_{n+a} \to \hat{f}_k \mathrm{e}^{-\mathrm{i}ka}, \qquad f_n \mathrm{e}^{\mathrm{i}pn} \to \hat{f}_{k+p},$$

- discrete differentiation:

$$f_n - f_{n-1} \to \hat{f}(k)(1-\mathrm{e}^{\mathrm{i}k}),$$

- product and convolution:

$$f_n g_n \to \frac{1}{L}\sum_\kappa \hat{f}_\kappa \hat{g}_{k-\kappa}, \qquad \sum_{m=0}^{L-1} f_m g_{n-m} \to \hat{f}_k\hat{g}_k,$$

- other identities:

$$\sum_{n=0}^{L-1} f_n g_n = \frac{1}{L}\sum_k \hat{f}_k \hat{g}_{2\pi-k}, \qquad \sum_{n=0}^{L-1} f_n g_{L-n} = \frac{1}{L}\sum_k \hat{f}_k\hat{g}_k.$$

Transformation table:

$$\sum_{k\neq 0} \frac{1}{1-\cos k} = \frac{L^2-1}{6}.$$

### 6.3. Generating function

Generating function is defined in context of the following transformation:

$$\tilde{f}(q) = \sum_{n=0}^{\infty} f_n q^n, \; q \in \mathbb{C},$$
$$f_n = \frac{1}{2\pi \mathrm{i} n!} \oint\limits_{q=0} \frac{\tilde{f}(q)\,\mathrm{d}q}{q^{n+1}} \equiv \frac{\tilde{f}^{(n)}(0)}{n!}, \; n \in \mathbb{Z}_+,$$

here $\tilde{f}(q)$ is the generating function of the sequence $\{f_n\}$; $f_n$ is assumed to be zero for $n < 0$.

Some properties:

- transform of unit function:
$$1 \to \frac{1}{1-q}, \qquad \delta_{n0} \to q^n,$$
- shift and dilatation:
$$f_{n-a} \to \tilde{f}(q) q^a, \; a \geqslant 0, \qquad f_n p^n \to \hat{f}(pq),$$
- multiplication on argument, differentiation, and summation:
$$n f_n \to q \frac{\mathrm{d}}{\mathrm{d}q} \tilde{f}(q), \qquad f_{n+1} - f_n \to \frac{\tilde{f}(q)(1-q) - f_0}{q}, \qquad f_n - f_{n-1} \to \tilde{f}(q)(1-q), \qquad \sum_{m=0}^{n} f_m \to \frac{\tilde{f}(q)}{1-q},$$
- convolution:
$$\sum_{m=0}^{n} f_m g_{n-m} \to \tilde{f}(q) \tilde{g}(q).$$

Tauberian theorems: Karamata proved for slow varying function $L$ that

$$\tilde{f}(q) \sim \frac{1}{(1-q)^{\mu}} L\left(\frac{1}{1-q}\right), \; q \to 1-0, \; \mu \geqslant 0 \implies \sum_{m=0}^{n} f_m \sim \frac{n^{\mu}}{\Gamma(\mu+1)} L(n), \; n \to \infty,$$

and if the sequence $\{f_n\}$ is monotonic then

$$f_n \sim \frac{n^{\mu-1}}{\Gamma(\mu)} L(n), \; n \to \infty.$$

In particular, if $\mu = 0$ then the statement becomes trivial: $\sum_{m=0}^{\infty} f_m = \tilde{f}(1)$.

## References


[1] W Feller, An introduction to probability theory and its applications (New York, Wiley, 1957, 1966); cited is russian edition of 1984.

[2] M V Fedoriuk, Asymptotics: Integrals and series (Moscow, Nauka, 1987); in russian.