Handbook on special functions

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September 8, 2023

§1. Elementary functions and some constants

Absolute value $|x|$ and **sign** function $\operatorname{sgn} x$.
Interpolation between linear and exponential function $[x]_q = \frac{q^x-1}{q-1}$, where $q \geqslant 1$, so that $[x]_1 = x$.
Minimum and **maximum** functions: $\min(a,b) \equiv a \wedge b$, $\max(a,b) \equiv a \vee b$.
Indicator $\mathcal{I}\{\mathrm{A}\} = \begin{cases} 1, & \text{if A,} \\ 0, & \text{if not A.} \end{cases}$
Divisibility $m|n$ is true iff m divides n, i.e. $\exists k \in \mathbb{Z}\ n = km$.

Euler's constant $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{-1} - \ln n\right) \approx 0.5772$.

§2. Exponential functions

Exponent $\exp x$ can be defined by the differential equation $y' = y$. An alternative definition is $\mathrm{e}^x = \lim_{n\to\infty}(1 + x/n)^n$. Series:

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots, \qquad |R_n(x)| < \frac{\max(\mathrm{e}^x, 1)|x|^{n+1}}{(n+1)!}.$$

Logarithm $\ln x$ is defined as the inverse exponent $x = \exp(\ln x)$. Series ($|x| < 1$):

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots, \qquad |R_n(x)| < \frac{|x|^{n+1}}{n+1}.$$

Trigonometric functions **sine** $\sin x$ and **cosine** $\cos x$ can be defined by the differential equation $y'' + y = 0$. They have period 2π. An alternative definition:

$$\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x, \qquad \sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}, \quad \cos x = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}.$$

Series:

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots, \qquad |R_{2n+1}(x)| < \frac{|x|^{2n+3}}{(2n+3)!},$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots, \qquad |R_{2n}(x)| < \frac{|x|^{2n+2}}{(2n+2)!}.$$

Other trigonometric functions are **tangent** $\tan x = \frac{\sin x}{\cos x}$ (its period is π), **cotangent** $\cot x = \frac{1}{\tan x}$, **secant** $\sec x = \frac{1}{\cos x}$, and **cosecant** $\csc x = \frac{1}{\sin x}$. Series ($|x| < \pi/2$):

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)|B_{2n}|}{(2n)!}x^{2n-1} = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \dots,$$

$$\cot x = \frac{1}{x} - \sum_{n=1} \frac{2^{2n}|B_{2n}|}{(2n)!}x^{2n-1} = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \dots,$$

$$\sec x = \sum_{n=0}^{\infty} \frac{|E_{2n}|}{(2n)!}x^{2n} = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \dots,$$

$$\csc x = \frac{1}{x} + \sum_{n=1} \frac{2\left(2^{2n-1}-1\right)|B_{2n}|}{(2n)!}x^{2n-1} = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \dots$$

Particular values:

x	0	$\frac{\pi}{12}$	$\frac{\pi}{10}$	$\frac{\pi}{6}$	$\frac{\pi}{5}$	$\frac{\pi}{4}$
$\sin x$	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$		$\frac{1}{\sqrt{2}}$
$\cos x$	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$		$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$
$\tan x$	0	$2-\sqrt{3}$		$\frac{1}{\sqrt{3}}$		1
$\frac{\pi}{2} - x$	0	$\frac{5\pi}{12}$	$\frac{2\pi}{5}$	$\frac{\pi}{3}$	$\frac{3\pi}{10}$	$\frac{\pi}{4}$

Inverse trigonometric functions have the following series expansions at $x = 0$:

$$\arcsin x = \sum_{n=0}^{\infty} \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}\frac{x^{2n+1}}{2n+1} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots, \qquad \arccos x = \frac{\pi}{2} - \arcsin x,$$

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots, \qquad \mathrm{arccot} x = \frac{\pi}{2} - \arctan x = \arctan\frac{1}{x}.$$

There is an extended version of arctangent:

$$\arctan(y, x) = \arg(x + \mathrm{i}y) \equiv \arctan\frac{y}{x} + \pi\,\mathcal{I}\{x < 0\}\,\mathrm{sgn}\,y.$$

Its properties:

$$\arctan(y,x)\in(-\pi,\pi),\quad \arctan(-y,x)=-\arctan(y,x),\quad \arctan(y,-x)=\pi-\arctan(y,x).$$

We will interpret $\arctan\frac{y}{x}$ as $\arctan(y,x)$ keeping the separate signs at x and y.

Hyperbolic sine $\sinh x$ and **cosine** $\cosh x$ can be defined by the differential equation $y''-y=0$. An alternative definition:

$$\mathrm{e}^x=\cosh x+\sinh x,\qquad \sinh x=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{2},\quad \cosh x=\frac{\mathrm{e}^x+\mathrm{e}^{-x}}{2}.$$

Series:

$$\sinh x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=x+\frac{1}{6}x^3+\frac{1}{120}x^5+\frac{1}{5040}x^7+\ldots,$$
$$\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}=1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\ldots$$

Other hyperbolic functions are **hyperbolic tangent** $\tanh x=\frac{\sinh x}{\cosh x}$ and **cotangent** $\coth x=\frac{1}{\tanh x}$. Series ($|x|<\pi/2$):

$$\tanh x=\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)B_{2n}}{(2n)!}x^{2n-1}=x-\frac{1}{3}x^3+\frac{2}{15}x^5-\frac{17}{315}x^7+\ldots,$$
$$\coth x=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}x^{2n-1}=\frac{1}{x}+\frac{1}{3}x-\frac{1}{45}x^3+\frac{2}{945}x^5-\ldots$$

Relation to trigonometric functions

$$\sinh \mathrm{i}x=\mathrm{i}\sin x,\quad \cosh \mathrm{i}x=\cos x,\quad \tanh \mathrm{i}x=\mathrm{i}\tan x,$$
$$\sin \mathrm{i}x=\mathrm{i}\sinh x,\quad \cos \mathrm{i}x=\cosh x,\quad \tan \mathrm{i}x=\mathrm{i}\tanh x.$$

Inverse hyperbolic functions can be reduced to the logarithmic form:

$$\operatorname{arcsinh} x=\ln\left(x+\sqrt{x^2+1}\right),\qquad \operatorname{arccosh} x=\ln\left(x+\sqrt{x^2-1}\right),\qquad \operatorname{arctanh} x=\frac{1}{2}\ln\frac{1+x}{1-x}.$$

Hyperbolic amplitude (gudermannian)

$$\operatorname{gd} x=2\arctan \mathrm{e}^x-\frac{\pi}{2},$$

so that

$$x=\ln\tan\left(\frac{\operatorname{gd} x}{2}+\frac{\pi}{4}\right),\qquad \sinh x=\tan\operatorname{gd} x,\quad \cosh x=\frac{1}{\cos\operatorname{gd} x},\quad \tanh x=\sin\operatorname{gd} x,\quad \tanh\frac{x}{2}=\tan\frac{\operatorname{gd} x}{2}.$$

§3. Algebraically defined functions

3.1. Lambert W function

Lambert W function is the solution of the equation

$$W(z)\mathrm{e}^{W(z)}=z.$$

Its branches are denoted by W_k so that the principal branch $W_0\equiv W$. The branch W_0 has the second order branching point at $z=-1/\mathrm{e}$ and the logarithmic branching point at infinity, the branch cut is real range $(-\infty,-1/\mathrm{e})$. The branches $W_{\pm1}$ have the second order branching point at $z=-1/\mathrm{e}$ and pair of logarithmic branching points at $z=0$ and infinity, the branch cut is real range $(-\infty,0)$. All other branches have pair of logarithmic branching points at $z=0$ and infinity with the branch cut taken to be real range $(-\infty,0)$. The branches numbering is of standard counterclockwise rule. The only exclusion is $(-1/\mathrm{e},0)$ subcut for the branches $W_{\pm1}$, this subcut connects these two branches (instead of W_0).

Series expansion for the principal branch:

$$W(z)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}z^n.$$

Series expansion at zero for nonprincipal branches or at infinity (the branches are distinguished by $\ln z$) is given by

$$W(z) = \frac{1-v}{u} + vF(u,v), \quad u = \frac{1}{\ln z}, \ v = \frac{\ln\ln z}{\ln z},$$

where F is regular function satisfying the equation

$$1 - uF(u,v) = \frac{1 - e^{-vF(u,v)}}{v},$$

so that $F(0,0) = 1$.

§4. Functions defined by generating function

4.1. Bernoulli numbers and polynomials

Bernoulli polynomials are defined as follows:

$$\frac{z\mathrm{e}^{xz}}{\mathrm{e}^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi.$$

The symbolic recurrence relation is

$$(B(x)+1)^n - B_n(x) = nx^{n-1}, \quad n \geqslant 0.$$

Bernoulli numbers are $B_n = B_n(0)$:

$$B_0 = 1, \ B_1 = -\frac{1}{2}, \ B_2 = \frac{1}{6}, \ B_4 = -\frac{1}{30}, \ B_6 = \frac{1}{42}, \qquad B_{2n+1} = 0, \ n \geqslant 1.$$

Miscellaneous relations:

$$|B_{2n}(x)| < |B_{2n}|, \qquad 0 < (-1)^{n+1}\frac{(2\pi)^{2n+1}}{2(2n+1)}B_{2n+1}(x) < \frac{1}{1-2^{-2n}}, \qquad 1 < (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n} < \frac{1}{1-2^{1-2n}}.$$

4.2. Euler numbers and polynomials

Euler polynomials are defined as follows:

$$\frac{2\mathrm{e}^{xz}}{\mathrm{e}^z + 1} = \sum_{n=0}^{\infty} E_n(x)\frac{z^n}{n!}, \quad |z| < \pi.$$

The symbolic recurrence relation is

$$(E(x)+1)^n + E_n(x) = 2x^n, \quad n \geqslant 0.$$

Euler numbers are $E_n = 2^n E_n(1/2)$:

$$E_0 = 1, \ E_2 = -1, \ E_4 = 5, \ E_6 = -61, \quad E_{2n+1} = 0.$$

§5. Functions defined by series

5.1. Riemann zeta function

Generalized **Riemann zeta** function is defined as

$$\zeta(z,q) = \sum_{n=0}^{\infty} \frac{1}{(q+n)^z},$$

Standard Riemann zeta function is $\zeta(z) \equiv \zeta(z,1)$.

Zeta function with noninteger q is analytic in the whole plane of z excluding simple pole at $z = 1$ with $c_{-1} = 1$ and $c_0 = -\psi(q)$. Standard zeta function has simple zeros at points $z = -2n, \ n \in \mathbb{N}$. Other zeros are nontrivial, they lie on

$\Re z = 1/2$ line (Riemann hypothesis) and they are uncountable. The importance of zeta function in number theory comes from the property

$$\zeta(z) = \prod_p \frac{1}{1 - p^{-z}},$$

where the product is over the prime numbers.

Integral representation:

$$\zeta(z, q) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{\mathrm{e}^{-qt} t^{z-1}}{1 - \mathrm{e}^{-t}}\, \mathrm{d}t,$$
$$\zeta(z, q) = -\frac{\Gamma(1-z)}{2\pi\mathrm{i}} \oint_C \frac{\mathrm{e}^{-qt}(-t)^{z-1}}{1 - \mathrm{e}^{-t}}\, \mathrm{d}t, \quad C = \{+\infty + \mathrm{i}\cdot 0,\ 0,\ +\infty - \mathrm{i}\cdot 0\}.$$

Some basic properties and particular values:

$$\zeta(z, q+1) = \zeta(z, q) - \frac{1}{q^z}, \qquad \zeta(z, 1/2) = (2^z - 1)\zeta(z),$$
$$\zeta(0, q) = \frac{1}{2} - q, \quad \zeta'(0, q) = \ln\Gamma(q) - \frac{\ln 2\pi}{2},$$
$$\zeta(z) = 2^z \pi^{z-1} \sin\frac{\pi z}{2} \Gamma(1-z)\zeta(1-z),$$
$$\zeta(0) = -\frac{1}{2},\ \zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(2n) = \frac{(2\pi)^{2n}|B_{2n}|}{2(2n)!}, \quad \zeta(1-2n) = -\frac{B_{2n}}{2n}.$$

Series expansion at $q = 1$:

$$\zeta(z, q) = \sum_{n=0}^\infty \frac{\zeta(n+z)\Gamma(n+z)}{\Gamma(z) n!}(1-q)^n.$$

Asymptotic expansion as $q \to \infty$:

$$\zeta(z, q+x) = \sum_{n=0}^\infty (-1)^n B_n(x) \frac{\Gamma(n+z-1)}{\Gamma(z) n!} \frac{1}{q^{n+z-1}} \stackrel{x=0}{=} \frac{1}{zq^{z-1}} + \frac{1}{2q^z} + \sum_{n=1}^\infty (-1)^{n+1} |B_{2n}| \frac{\Gamma(2n+z-1)}{\Gamma(z)(2n)!} \frac{1}{q^{2n+z-1}}.$$

5.2. Polylogarithm

Polylogarithm is defined as

$$\mathrm{Li}(s, z) = \sum_{n=1}^\infty \frac{z^n}{n^s}.$$

Polylogarithm of index 2 is called **dilogarithm**. The integral representation is

$$\mathrm{Li}(s, z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}\,\mathrm{d}t}{e^t - z}.$$

Polylogarithm is analytic in the whole plane of z excluding nontrivial branching points at $z = 1$ and $z = \infty$ with the branch cut taken to be the real range $(1, +\infty)$ and another pair of logarithmic branching points at $z = 0$ and $z = \infty$ with the branch cut taken to be the negative real axis, note that the last pair of branching points $(-\infty, 0)$ is absent in the principal Riemann plane of the first pair of branching points $(1, +\infty)$. The analytic continuation from the unit disk is provided by the formula

$$\mathrm{Li}(s, z) + \mathrm{e}^{s\pi\mathrm{i}} \mathrm{Li}\left(s, z^{-1}\right) = \frac{(2\pi)^s}{\Gamma(s)} \mathrm{e}^{s\pi\mathrm{i}/2} \zeta\left(1 - s, \frac{\ln z}{2\pi\mathrm{i}}\right),$$

the analytic continuation across the branch cut $(1, +\infty)$ is provided by the formula

$$\mathrm{Li}(s, z) - \mathrm{Li}\left(s, z\mathrm{e}^{2\pi\mathrm{i}}\right) = \frac{2\pi\mathrm{i}}{\Gamma(s)} \ln^{s-1} z,$$

and the analytic continuation across the branch cut $(-\infty, 0)$ is determined as for the ordinary logarithm function.

Some properties and particular values:

$$\frac{\partial \mathrm{Li}(s, z)}{\partial z} = \frac{\mathrm{Li}(s-1, z)}{z}, \qquad \mathrm{Li}\left(s, z^n\right) = n^{s-1} \sum_{k=1}^n \mathrm{Li}\left(s, z\mathrm{e}^{2\pi\mathrm{i}k/n}\right),$$
$$\mathrm{Li}(s, 1) = \zeta(s), \quad \mathrm{Li}(0, z) = \frac{z}{1-z}, \quad \mathrm{Li}(1, z) = -\ln(1-z).$$

Series expansion in powers of $\ln z$ at point $z=1$:

$$\mathrm{Li}(s,z)=\Gamma(1-s)(-\ln z)^{s-1}+\sum_{n=0}^{\infty}\frac{\zeta(s-n)}{n!}(\ln z)^n,$$

asymptotic expansion as $z\to-\infty$:

$$\mathrm{Li}(s,z)\sim\sum_{n=0}^{\infty}\left|\left(1-2^{1-2n}\right)B_{2n}\right|\frac{(2\pi)^{2n}\Gamma(2n-s)}{\Gamma(s)\Gamma(1-s)(2n)!}\,\frac{1}{\ln(-z)^{2n-s}}.$$

5.3. Lerch phi function

Lerch phi function generalizes Riemann zeta function and polylogarithm:

$$\Phi(z,s,q)=\sum_{n=0}^{\infty}\frac{z^n}{(q+n)^s},$$

so that $\zeta(s,q)=\Phi(1,s,q)$ and $\mathrm{Li}(s,z)=z\Phi(z,s,1)$. Its analytic properties is the same as of polylogarithm.

Integral representation:

$$\Phi(z,s,q)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{\mathrm{e}^{-qt}t^{s-1}}{1-z\mathrm{e}^{-t}}\,\mathrm{d}t.$$

Some properties:

$$\Phi(z,s,q)=z\Phi(z,s,q+1)+q^{-s}.$$

Series expansion in powers of $\ln z$ at point $z=1$:

$$z^q\Phi(z,s,q)=\frac{\Gamma(1-s)}{(-\ln z)^{1-s}}+\sum_{n=0}^{\infty}\frac{\zeta(s-n,q)}{n!}(\ln z)^n.$$

For efficient computations with large q the following formula can be used

$$\Phi(z,s,q)=\frac{\Gamma(1-s,-q\ln z)}{z^q\,(-\ln z)^{1-s}}+\sum_{n=0}^{\infty}\frac{(s)_n}{n!}\frac{R_n(z)}{q^{n+s}},\quad R_n(z)=\left(-z\frac{\mathrm{d}}{\mathrm{d}z}\right)^n\left(\frac{1}{\ln z}+\frac{1}{1-z}\right),$$

here R_n is analytic at 1 despite the two addends are diverging progressively with n, thus requiring use of series around 1 computed recursively. Note that $R_n(z)\equiv n!/(\ln z)^{n+1}+(-1)^n\mathrm{Li}(-n,z)$ for $n>0$.

§6. Functions defined by integral

6.1. Exponential integral and related functions

Exponential, sine and cosine integrals have different definitions. We use Maple definitions:

$$\mathrm{Ei}(x)=\int_{-\infty}^{x}\frac{\mathrm{e}^t}{t}\,\mathrm{d}t,\quad \mathrm{Si}(x)=\int_0^x\frac{\sin t}{t}\,\mathrm{d}t,\quad \mathrm{Ci}(x)=-\int_x^{+\infty}\frac{\cos t}{t}\,\mathrm{d}t.$$

Series expansions:

$$\mathrm{Ei}(x)=\gamma+\ln x+x+\frac{x^2}{4}+\frac{x^3}{18}+\ldots+\frac{x^n}{n\,n!}+\ldots,$$
$$\mathrm{Si}(x)=x-\frac{x^3}{18}+\frac{x^5}{600}-\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}+\ldots,$$
$$\mathrm{Ci}(x)=\gamma+\ln x-\frac{x^2}{4}+\frac{x^4}{96}+\ldots+\frac{(-1)^n x^{2n}}{2n(2n)!}+\ldots.$$

Logarithmic integral can be reduced to Ei:

$$\mathrm{Li}(x)=\int_0^x\frac{\mathrm{d}t}{\ln t}\equiv\mathrm{Ei}(\ln x).$$

6.2. Error function and Fresnel integrals

Error function has different definitions. We use Maple's definition:

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t,\qquad \operatorname{erfc}(x)=1-\operatorname{erf}(x),\qquad \operatorname{erfi}(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t\equiv -\mathrm{i}\operatorname{erf}(\mathrm{i}x).$$

Series expansion:

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{3}+\frac{x^5}{10}-\ldots+\frac{(-1)^n x^{2n+1}}{(2n+1)n!}+\ldots\right),$$

asymptotic expansion as $x\to+\infty$:

$$\operatorname{erf}(x)\sim 1-\frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2}\left(\frac{1}{x}-\frac{1}{2x^3}+\frac{3}{4x^5}-\ldots+\frac{(-1)^n(2n-1)!!}{2^n x^{2n+1}}+\ldots\right).$$

Inequalities: $\operatorname{erfc}(x)\leqslant \mathrm{e}^{-x^2}$.

Fresnel sine and cosine integrals have also different definitions. We use Maple's definition:

$$\begin{pmatrix}S\\C\end{pmatrix}(x)=\int_0^x\begin{pmatrix}\sin\\\cos\end{pmatrix}\left(\frac{\pi}{2}t^2\right)\,\mathrm{d}t.$$

Asymptotic expansions as $x\to+\infty$:

$$\begin{aligned}S(x)&=\frac{1}{2}-\frac{1}{\pi x}\cos\left(\frac{\pi}{2}x^2\right)-\frac{1}{\pi^2x^3}\sin\left(\frac{\pi}{2}x^2\right)+\ldots,\\ C(x)&=\frac{1}{2}+\frac{1}{\pi x}\sin\left(\frac{\pi}{2}x^2\right)-\frac{1}{\pi^2x^3}\cos\left(\frac{\pi}{2}x^2\right)-\ldots\end{aligned}$$

§7. Elliptic functions

Elliptic function is a doubly periodic meromorphic function. The minimal periods are denoted by 2ω and $2\omega'$. The order of an elliptic function is the number of poles in the elementary cell with their multiplicity counted. Elliptic functions have many remarkable properties. One of them is that an elliptic function is fully determined (plus arbitrary constant) by only its periods, poles and principal parts in these poles. Any two elliptic functions with identical periods are connected by an algebraic (polynomial) equation. The same statement is valid for an elliptic function and its derivative.

7.1. Elliptic integrals

Elliptic integrals of the first, second, and third kinds are defined as follows ($z\equiv\sin\phi$):

$$\begin{aligned}F(\phi,k)&=\int_0^\phi\frac{\mathrm{d}\alpha}{\sqrt{1-k^2\sin^2\alpha}} &&\equiv F(z,k)=\int_0^z\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}},\\ E(\phi,k)&=\int_0^\phi\sqrt{1-k^2\sin^2\alpha}\,\mathrm{d}\alpha &&\equiv E(z,k)=\int_0^z\sqrt{\frac{(1-k^2x^2)}{(1-x^2)}}\,\mathrm{d}x,\\ \Pi(\phi,\nu,k)&=\int_0^\phi\frac{\mathrm{d}\alpha}{(1-\nu\sin^2\alpha)\sqrt{1-k^2\sin^2\alpha}} &&\equiv \Pi(z,\nu,k)=\int_0^z\frac{\mathrm{d}x}{(1-\nu x^2)\sqrt{(1-x^2)(1-k^2x^2)}}.\end{aligned}$$

Here k is the modulus and $k'=\sqrt{1-k^2}$ is the complementary modulus. Elliptic integrals of the complementary modulus are called complementary elliptic integrals and denoted by prime, e.g. $\mathrm{K}'(k)\equiv\mathrm{K}(k')$. Limiting cases:

$$F(\phi,0)=E(\phi,0)=\phi,\quad F(\phi,1)=\ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right),\quad E(\phi,1)=\sin\phi.$$

Complete elliptic integrals are $\mathrm{K}(k)=F(\pi/2,k)$, $\mathrm{E}(k)=E(\pi/2,k)$ and $\Pi(\nu,k)=\Pi(\pi/2,\nu,k)$. It should be noted that the third integral can be reduced sometimes to the first two. The first and second integrals can be expressed via hypergeometric function as follows:

$$\mathrm{K}(k)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;k^2\right),\qquad \mathrm{E}(k)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;k^2\right).$$

This representation suggests that the complete elliptic integrals should be considered as functions of the argument $m \equiv k^2$ instead of k. Both $\mathrm{K}(m)$ and $\mathrm{E}(m)$ are analytical functions except two branching logarithmic points $m=1$ and $m=\infty$. The branch cut is usually the real range $(1,+\infty)$.

Series:

$$\mathrm{K}(k)=\frac{\pi}{2}\left(1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots+\left(\frac{(2n-1)!!}{(2n)!!}\right)^2k^{2n}+\ldots\right),$$
$$\mathrm{E}(k)=\frac{\pi}{2}\left(1-\frac{1}{4}k^2-\frac{3}{64}k^4-\ldots-\left(\frac{(2n-1)!!}{(2n)!!}\right)^2\frac{k^{2n}}{2n-1}-\ldots\right),$$
$$\Pi(\nu,k)=\frac{\pi}{2}\left(\frac{1}{\sqrt{1-\nu}}+\frac{1}{2\sqrt{1-\nu}\left(1+\sqrt{1-\nu}\right)}k^2+\ldots+\left(\frac{(2n-1)!!}{(2n)!!}\right)^2F\left(1,n+\frac{1}{2};n+1;\nu\right)k^{2n}+\ldots\right).$$

Series expansion at $k=1$:

$$\mathrm{K}(k)=\frac{2}{\pi}\mathrm{K}(k')\ln\frac{4}{k'}-\frac{1}{4}k'^2-\frac{21}{128}k'^4-\ldots-c_n\left(\frac{(2n-1)!!}{(2n)!!}\right)^2k'^{2n}-\ldots,$$
$$\mathrm{E}(k)=1+\frac{2}{\pi}\left[\mathrm{K}(k')-\mathrm{E}(k')\right]\ln\frac{4}{k'}-\frac{1}{4}k'^2-\frac{13}{64}k'^4-\ldots-c_n\left(\frac{2n}{2n-1}\right)\left(\frac{(2n-1)!!}{(2n)!!}\right)^2k'^{2n}-\ldots,$$
$$c_n=\sum_{m=1}^{n}\frac{1}{m(2m-1)}=\frac{\ln 2}{2}+\frac{\psi(n+1/2)}{4}-\frac{\psi(n+1)}{4}.$$

Differentiation formulas:

$$\frac{\partial F(\phi,k)}{\partial k}=\frac{1}{k'^2}\left(\frac{E(\phi,k)-k'^2F(\phi,k)}{k}-\frac{k\sin\phi\cos\phi}{\sqrt{1-k^2\sin^2\phi}}\right),\qquad\frac{\partial E(\phi,k)}{\partial k}=\frac{E(\phi,k)-F(\phi,k)}{k},$$
$$\frac{\mathrm{dK}}{\mathrm{d}k}=\frac{1}{k}\left(\frac{\mathrm{E}}{k'^2}-\mathrm{K}\right),\qquad\frac{\mathrm{dE}}{\mathrm{d}k}=\frac{\mathrm{E}-\mathrm{K}}{k}.$$

Transformation formulas provide analytic continuation and simplify calculation of the complete elliptic integrals:

$$\mathrm{K}\left(\frac{1}{k}\pm\mathrm{i}0\right)=k\left[\mathrm{K}(k)\pm\mathrm{i}\mathrm{K}(k')\right],\qquad\mathrm{K}\left(\mathrm{i}\frac{k}{k'}\right)=k'\mathrm{K}(k),$$
$$\mathrm{E}\left(\frac{1}{k}\pm\mathrm{i}0\right)=\frac{1}{k}\left\{\left[\mathrm{E}(k)-k'^2\mathrm{K}(k)\right]\mp\mathrm{i}\left[\mathrm{E}(k')-k^2\mathrm{K}(k')\right]\right\},\qquad\mathrm{E}\left(\mathrm{i}\frac{k}{k'}\right)=\frac{1}{k'}\mathrm{E}(k),$$

the following transformations are mutually inverse:

$$\kappa=\frac{1-k'}{1+k'},\qquad k=\frac{2\sqrt{\kappa}}{1+\kappa},$$
$$\mathrm{K}\left(\frac{1-k'}{1+k'}\right)=\frac{1+k'}{2}\mathrm{K}(k),\qquad\mathrm{E}\left(\frac{1-k'}{1+k'}\right)=\frac{\mathrm{E}(k)+k'\mathrm{K}(k)}{1+k'},$$
$$\mathrm{K}\left(\frac{2\sqrt{\kappa}}{1+\kappa}\right)=(1+\kappa)\mathrm{K}(\kappa),\qquad\mathrm{E}\left(\frac{2\sqrt{\kappa}}{1+\kappa}\right)=\frac{2E(\kappa)-\kappa'^2\mathrm{K}(\kappa)}{1+\kappa}.$$

The third elliptic integral $\Pi(\nu,k)$ can be transformed to the interval $\nu=[-k,k]$ using the identity

$$\Pi(ka,k)+\Pi(k/a,k)-\mathrm{K}(k)=\frac{\pi}{2}\frac{1}{\sqrt{(1-ka)(1-k/a)}}\mathcal{I}\left\{(1-ka)(1-k/a)>0\right\}.$$

Miscellaneous relations:

$$\mathrm{KE}'+\mathrm{K}'\mathrm{E}-\mathrm{KK}'=\pi/2.$$

7.2. Jacobi elliptic functions

Jacobi amplitude is the inverse of the first kind elliptic integral: $F(\operatorname{am}z,k)=z$. Note that modulus is usually omitted in Jacobi elliptic functions. **Elliptic sine** and **cosine** are $\operatorname{sn}z=\sin\operatorname{am}z$ and $\operatorname{cn}z=\cos\operatorname{am}z$. **Delta of amplitude** is $\operatorname{dn}z=\frac{\mathrm{d}\operatorname{am}z}{\mathrm{d}z}$ and **Jacobi zeta** function is $\operatorname{zn}z=E(\operatorname{am}z,k)-z\mathrm{E}/\mathrm{K}$.

Limiting cases:

$$k=0:\quad \operatorname{am}z=z,\ \operatorname{sn}z=\sin z,\ \operatorname{cn}z=\cos z,\ \operatorname{dn}z=1,\ \operatorname{zn}z=0\ (\text{as }(k^2/4)\sin 2z),$$
$$k=1:\quad \operatorname{am}z=\operatorname{gd}z,\ \operatorname{sn}z=\operatorname{zn}z=\tanh z,\ \operatorname{cn}z=\operatorname{dn}z=1/\cosh z.$$

Elliptic sine, cosine and delta of amplitude are the second order elliptic functions with two simple poles in unit cell.

	2ω	$2\omega'$	zeros	poles	residues
sn	4K	$2\mathrm{i}\mathrm{K}'$	$2m\mathrm{K}+2n\mathrm{i}\mathrm{K}'$	$2m\mathrm{K}+(2n+1)\mathrm{i}\mathrm{K}'$	$(-1)^m/k$
cn	4K	$2\mathrm{K}+2\mathrm{i}\mathrm{K}'$	$(2m+1)\mathrm{K}+2n\mathrm{i}\mathrm{K}'$	the same	$(-1)^{m+n}/(\mathrm{i}k)$
dn	2K	$4\mathrm{i}\mathrm{K}'$	$(2m+1)\mathrm{K}+(2n+1)\mathrm{i}\mathrm{K}'$	the same	$(-1)^n/\mathrm{i}$

Their transformation properties:

	$\mathrm{i}z$	kz	$z+\mathrm{K}$	$z+\mathrm{i}\mathrm{K}'$	$z+\mathrm{K}+\mathrm{i}\mathrm{K}'$	$z+2\mathrm{K}$	$z+2\mathrm{i}\mathrm{K}'$
sn	$\frac{\mathrm{i}\,\mathrm{sn}(z,k')}{\mathrm{cn}(z,k')}$	$\frac{1}{k}\,\mathrm{sn}\left(z,\frac{1}{k}\right)$	$\frac{\mathrm{cn}\,z}{\mathrm{dn}\,z}$	$\frac{1}{k\,\mathrm{sn}\,z}$	$\frac{\mathrm{dn}\,z}{k\,\mathrm{cn}\,z}$	$-\,\mathrm{sn}\,z$	$\mathrm{sn}\,z$
cn	$\frac{1}{\mathrm{cn}(z,k')}$	$\mathrm{dn}\left(z,\frac{1}{k}\right)$	$-\frac{k'\,\mathrm{sn}\,z}{\mathrm{dn}\,z}$	$-\frac{\mathrm{i}\,\mathrm{dn}\,z}{k\,\mathrm{sn}\,z}$	$-\frac{\mathrm{i}k'}{k\,\mathrm{cn}\,z}$	$-\,\mathrm{cn}\,z$	$-\,\mathrm{cn}\,z$
dn	$\frac{\mathrm{dn}(z,k')}{\mathrm{cn}(z,k')}$	$\mathrm{cn}\left(z,\frac{1}{k}\right)$	$\frac{k'}{\mathrm{dn}\,z}$	$-\frac{\mathrm{i}\,\mathrm{cn}\,z}{\mathrm{sn}\,z}$	$\frac{\mathrm{i}k'\,\mathrm{sn}\,z}{\mathrm{cn}\,z}$	$\mathrm{dn}\,z$	$-\,\mathrm{dn}\,z$

Particular values:

	0	$\mathrm{K}/2$	$\mathrm{i}\mathrm{K}'/2$	$(\mathrm{K}+\mathrm{i}\mathrm{K}')/2$
sn	0	$\frac{1}{\sqrt{1+k'}}$	$\frac{\mathrm{i}}{\sqrt{k}}$	$\frac{\sqrt{1+k}+\mathrm{i}\sqrt{1-k}}{\sqrt{2k}}$
cn	1	$\sqrt{\frac{k'}{1+k'}}$	$\sqrt{\frac{1+k}{k}}$	$\frac{(1-\mathrm{i})\sqrt{k'}}{\sqrt{2k}}$
dn	1	$\sqrt{k'}$	$\sqrt{1+k}$	$\frac{\sqrt{k'}\left(\sqrt{1+k'}-\mathrm{i}\sqrt{1-k'}\right)}{\sqrt{2}}$

Amplitude has period $4\mathrm{i}\mathrm{K}'$ and branching logarithmic points at $2m\mathrm{K}+(2n+1)\mathrm{i}\mathrm{K}'$, also $\mathrm{am}(z+2\mathrm{K})=\pi+\mathrm{am}\,z$ and $\mathrm{am}(z+2\mathrm{i}\mathrm{K}')=\pi-\mathrm{am}\,z$. Zeta function is meromorphic with period 2K.

Some identities:

$$\mathrm{sn}^2\,z+\mathrm{cn}^2\,z=1,\qquad \mathrm{dn}^2\,z=1-k^2\,\mathrm{sn}^2\,z=k'^2+k^2\,\mathrm{cn}^2\,z,$$

$$\mathrm{sn}(z_1\pm z_2)=\frac{s_1c_2d_2\pm s_2c_1d_1}{1-k^2s_1^2s_2^2},\qquad \mathrm{cn}(z_1\pm z_2)=\frac{c_1c_2\mp s_1s_2d_1d_2}{1-k^2s_1^2s_2^2},\qquad \mathrm{dn}(z_1\pm z_2)=\frac{d_1d_2\mp k^2s_1s_2c_1c_2}{1-k^2s_1^2s_2^2},$$

$$\mathrm{sn}^2\,\frac{z}{2}=\frac{1-\mathrm{cn}\,z}{1+\mathrm{dn}\,z},\qquad \mathrm{cn}^2\,\frac{z}{2}=\frac{\mathrm{cn}\,z+\mathrm{dn}\,z}{1+\mathrm{dn}\,z},\qquad \mathrm{dn}^2\,\frac{z}{2}=\frac{\mathrm{cn}\,z+\mathrm{dn}\,z}{1+\mathrm{cn}\,z}.$$

Differentiation:

$$(\mathrm{am}\,z)'=\mathrm{dn}\,z,\ (\mathrm{sn}\,z)'=\mathrm{cn}\,z\,\mathrm{dn}\,z,\ (\mathrm{cn}\,z)'=-\,\mathrm{sn}\,z\,\mathrm{dn}\,z,\ (\mathrm{dn}\,z)'=-k^2\,\mathrm{sn}\,z\,\mathrm{cn}\,z,\ (\mathrm{zn}\,z)'=\mathrm{dn}^2\,z-\mathrm{E}/\mathrm{K}.$$

Differential equations: $\mathrm{sn}\,z$ satisfies the equations $s''=(1-s^2)(1-k^2s^2)$ and $s''+(1+k^2)s=2k^2s^3$, $\mathrm{am}\,z$ satisfies the equation $\phi''+(k^2/2)\sin 2\phi=0$.

7.3. Weierstrass elliptic functions

Weierstrass function $\wp(z)$ is defined as the second order elliptic function with one pole in unit cell at point $z=0$ with the principal part z^{-2} and $\wp(z)-z^{-2}$ equal zero at this point. Expansion in full partial fractions yields

$$\wp(z)=\frac{1}{z^2}+\sum_{m,n}\left[\frac{1}{(z-2\omega_{mn})^2}-\frac{1}{(2\omega_{mn})^2}\right],$$

where $\omega_{mn}=m\omega+n\omega'$, 2ω and $2\omega'$ are periods, and all double sums are over $\mathbb{Z}^2$ excluding $m=n=0$ in this subsection. $\wp$ is even function. The first derivative of the Weierstrass function is also of great importance due to the following theorem: Any elliptic function can be presented in the form $R_1\left[\wp(z)\right]+\wp'(z)R_2\left[\wp(z)\right]$, where $R_{1,2}$ are rational functions. **Weierstrass zeta** function $\zeta(z)$ is defined as an odd function satisfying equation $\zeta'=-\wp$. **Weierstrass sigma** function $\sigma(z)$ is defined as an entire function satisfying equation $\sigma'/\sigma=\zeta$. Both ζ and σ are not elliptic functions. σ is odd function. ζ has simple poles and σ has simple zeros at points $2\omega_{mn}$.

Some notations and definitions:

$$\omega_1=\omega,\quad \omega_2=-\omega-\omega',\quad \omega_3=\omega',\quad \eta=\zeta(\omega),\quad \eta'=\zeta(\omega'),\quad \eta_i=\zeta(\omega_i),\quad \sigma_i(z)=\frac{\sigma(z+\omega_i)}{\sigma(\omega_i)}\mathrm{e}^{-z\eta_i}.$$

Alternative definition of the Weierstrass function follows from the normal elliptic Weierstrass integral of the first kind:

$$z=\int_{\wp(z)}^{+\infty}\frac{\mathrm{d}w}{\sqrt{4w^3-g_2w-g_3}}=\int_{\wp(z)}^{+\infty}\frac{\mathrm{d}w}{\sqrt{4(w-e_1)(w-e_2)(w-e_3)}},$$

or the corresponding differential equation $w'^2 = 4w^3 - g_2 w - g_3$. Periods, invariants $g_{2,3}$ and roots $e_{1,2,3}$ are connected by the relations:

$$e_1 + e_2 + e_3 = 0, \quad e_1e_2 + e_2e_3 + e_3e_1 = -g_2/4, \quad e_1e_2e_3 = g_3/4, \qquad e_i = \wp(\omega_i),$$
$$g_2 = 60\sum_{m,n}\frac{1}{(2\omega_{mn})^4}, \quad g_3 = 140\sum_{m,n}\frac{1}{(2\omega_{mn})^6}.$$

The combination $\Delta = g_2^3 - 27g_3^2 \equiv [4(e_1-e_2)(e_2-e_3)(e_3-e_1)]^2$ is the discriminant. If $\Delta > 0$ then $e_{1,2,3}$ are real, ω is real, and ω' is imaginary. In this case

$$\omega = \int_{e_1}^{+\infty}\frac{\mathrm{d}w}{\sqrt{4w^3 - g_2w - g_3}}, \qquad \omega' = \mathrm{i}\int_{-\infty}^{e_3}\frac{\mathrm{d}w}{\sqrt{-4w^3 + g_2w + g_3}}.$$

If $\Delta < 0$ then e_2 is real, $e_{1,3} = \alpha \pm \mathrm{i}\beta$, and ω, ω' are complex conjugated so that $\omega_1 = \omega + \omega'$ and $\omega_2 = \omega - \omega'$ are determined from

$$\omega_1 = \int_{e_2}^{+\infty}\frac{\mathrm{d}w}{\sqrt{4w^3 - g_2w - g_3}}, \qquad \omega_2 = \mathrm{i}\int_{-\infty}^{e_2}\frac{\mathrm{d}w}{\sqrt{-4w^3 + g_2w + g_3}}.$$

If $\Delta = 0$ then:

$$1)\ e_1 = e_2 = a,\ e_3 = -2a, \qquad \omega = \infty,\ \omega' = \frac{\mathrm{i}\pi}{\sqrt{12a}}, \qquad \wp(z) = a + 3a\sinh^{-2}\left(z\sqrt{3a}\right),$$
$$2)\ e_1 = 2a,\ e_2 = e_3 = -a, \qquad \omega = \frac{\pi}{\sqrt{12a}},\ \omega' = \mathrm{i}\infty, \qquad \wp(z) = -a + 3a\sin^{-2}\left(z\sqrt{3a}\right),$$
$$3)\ e_1 = e_2 = e_3 = 0, \qquad \omega = \infty,\ \omega' = \mathrm{i}\infty, \qquad \wp(z) = z^{-2},\ \zeta(z) = z^{-1},\ \sigma(z) = z.$$

Transformations:

$$\wp(z_1 + z_2) = -\wp(z_1) - \wp(z_2) + \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2,$$
$$\zeta(z_1 + z_2) = \zeta(z_1) + \zeta(z_2) + \frac{1}{2}\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)},$$
$$\sigma(z_1 + z_2)\sigma(z_1 - z_2) = -\sigma^2(z_1)\sigma^2(z_2)\left(\wp(z_1) - \wp(z_2)\right),$$
$$\zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i,$$
$$\sigma(z + 2\omega_i) = -\sigma(z)\mathrm{e}^{2\eta_i(z+\omega_i)}, \quad \sigma_i(z + 2\omega_j) = (-1)^{\delta_{ij}}\sigma_i(z)\mathrm{e}^{2\eta_j(z+\omega_j)}.$$

Series:

$$\wp(z) = \frac{1}{z^2} + \frac{g_2}{20}z^2 + \frac{g_3}{28}z^4 + \frac{g_2^2}{1200}z^6 + \dots,$$
$$\sigma(z) = z - \frac{g_2}{240}z^5 - \frac{g_3}{840}z^7 - \dots$$

Relations to Jacobi elliptic functions:

$$k^2 = \frac{e_{23}}{e_{13}}, \quad \mathrm{K} = \omega\sqrt{e_{13}}, \quad \mathrm{K}' = -\mathrm{i}\omega'\sqrt{e_{13}}, \quad \mathrm{E} = \frac{e_1\omega + \eta}{\sqrt{e_{13}}}, \quad \mathrm{E}' = \mathrm{i}\frac{e_3\omega' + \eta'}{\sqrt{e_{13}}},$$
$$u = z\sqrt{e_{13}}, \quad \operatorname{sn} u = \sqrt{\frac{e_{13}}{\wp(z) - e_3}}, \quad \operatorname{cn} u = \sqrt{\frac{\wp(z) - e_1}{\wp(z) - e_3}}, \quad \operatorname{dn} u = \sqrt{\frac{\wp(z) - e_2}{\wp(z) - e_3}},$$

where $e_{ij} = e_i - e_j$ and all the square roots are uniquely determined by

$$\sqrt{\wp(z) - e_i} = \frac{\sigma_i(z)}{\sigma(z)}, \quad \sqrt{e_{ij}} = \frac{\sigma_j(\omega_i)}{\sigma(\omega_i)}.$$

Miscellaneous identities and equations:

$$\sigma(z) = z\exp\int_0^z\left(\zeta(t) - \frac{1}{t}\right)\mathrm{d}t, \qquad \wp'' = 6\wp^2 - g_2/2.$$

7.4. Jacobi theta functions

Generally a theta function is any quasi doubly periodic entire function that is

$$\theta(z+1)=\theta(z), \quad \theta(z+\tau)=\mathrm{e}^{-2\pi \mathrm{i} kz}\theta(z),\ \Im\tau>0.$$

Jacobi theta functions are defined as follows:

$$\begin{aligned}
\theta_1(z,\tau) &= \sum_{n\in\mathbb{Z}} \exp\left[\mathrm{i}\pi(n+1/2)+\mathrm{i}\pi(n+1/2)^2\tau+\mathrm{i}\pi(2n+1)z\right] &&= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)\pi z,\\
\theta_2(z,\tau) &= \sum_{n\in\mathbb{Z}} \exp\left[\mathrm{i}\pi(n+1/2)^2\tau+\mathrm{i}\pi(2n+1)z\right] &&= 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)\pi z,\\
\theta_3(z,\tau) &= \sum_{n\in\mathbb{Z}} \exp\left[\mathrm{i}\pi n^2\tau+\mathrm{i}\pi 2nz\right] &&= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2n\pi z,\\
\theta_4(z,\tau) &= \sum_{n\in\mathbb{Z}} \exp\left[\mathrm{i}\pi n+\mathrm{i}\pi n^2\tau+\mathrm{i}\pi 2nz\right] &&= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2n\pi z,
\end{aligned}$$

where $q\equiv \mathrm{e}^{\mathrm{i}\pi\tau}$ is the nome, $|q|<1$. Values at zero are denoted as θ_i without arguments that is $\theta_i=\theta_i(0,\tau)$.

Properties and transformations:

	period	parity	zeros	$z+1/2$	$z+\tau/2$	$z+1/2+\tau/2$	$z+1$	$z+\tau$	$z+1+\tau$
θ_1	1	odd	$m+n\tau$	θ_2	$\mathrm{i}B\theta_4$	$B\theta_3$	$-\theta_1$	$-A\theta_1$	$A\theta_1$
θ_2	1	even	$\left(m+\frac{1}{2}\right)+n\tau$	$-\theta_1$	$B\theta_3$	$-\mathrm{i}B\theta_4$	$-\theta_2$	$A\theta_2$	$-A\theta_2$
θ_3	2	even	$\left(m+\frac{1}{2}\right)+\left(n+\frac{1}{2}\right)\tau$	θ_4	$B\theta_2$	$\mathrm{i}B\theta_1$	θ_3	$A\theta_3$	$A\theta_3$
θ_4	2	even	$m+\left(n+\frac{1}{2}\right)\tau$	θ_3	$\mathrm{i}B\theta_1$	$B\theta_2$	θ_4	$-A\theta_4$	$-A\theta_4$

where θ_i means $\theta_i(z,\tau)$, $A=\mathrm{e}^{-\mathrm{i}\pi(2z+\tau)}$ and $B=\mathrm{e}^{-\mathrm{i}\pi(z+\tau/4)}$.

Relations to Jacobi elliptic functions are based on the elliptic modulus function $k^2(q)=(\theta_2/\theta_3)^4$ and its inverse nome function $q(k^2)=\exp(-\pi\mathrm{K}'/\mathrm{K})$. They can be approximated as follows:

$$k^2\approx 16q\frac{1+4q^2}{1+8q+24q^2}, \qquad q=\varepsilon+2\varepsilon^5+15\varepsilon^9+150\varepsilon^{13}+\dots, \qquad \varepsilon=\frac{1}{2}\frac{1-\sqrt{k'}}{1+\sqrt{k'}}.$$

Elliptic integrals and Jacobi elliptic functions:

$$\mathrm{K}=\frac{1}{2}\pi\theta_3^2, \qquad \mathrm{K}'=-\mathrm{i}\tau\mathrm{K}, \qquad \mathrm{E}=-\frac{1}{2\pi}\frac{\theta_2''}{\theta_2}\frac{1}{\theta_3^2}, \qquad \mathrm{E}'=\frac{1}{\theta_3^2}\left(1-\frac{\mathrm{i}\tau}{2\pi}\frac{\theta_4''}{\theta_4}\right),$$
$$\mathrm{sn}(2\mathrm{K}z)=\frac{\theta_3}{\theta_2}\frac{\theta_1(z)}{\theta_4(z)}, \qquad \mathrm{cn}(2\mathrm{K}z)=\frac{\theta_4}{\theta_2}\frac{\theta_2(z)}{\theta_4(z)}, \qquad \mathrm{dn}(2\mathrm{K}z)=\frac{\theta_4}{\theta_3}\frac{\theta_3(z)}{\theta_4(z)}.$$

Relations to Weierstrass elliptic functions ($\tau=\omega'/\omega$):

$$\sqrt{e_{13}}=\frac{\pi\theta_3^2}{2\omega}, \qquad e_1=\frac{\pi^2\left(\theta_3^4+\theta_4^4\right)}{12\omega^2}, \qquad e_2=\frac{\pi^2\left(\theta_2^4-\theta_4^4\right)}{12\omega^2}, \qquad e_3=-\frac{\pi^2\left(\theta_2^4+\theta_3^4\right)}{12\omega^2}, \qquad \eta=-\frac{\pi^2\theta_1'''}{12\omega\theta_1'},$$
$$\wp(2\omega z)=e_i+\left(\frac{\pi\theta_1'}{2\omega\theta_{i+1}}\right)^2\left(\frac{\theta_{i+1}(z)}{\theta_1(z)}\right)^2,\ i=1,2,3, \qquad \wp'(2\omega z)=-\left(\frac{\pi\theta_1'}{2\omega}\right)^3\frac{\theta_1(2z)}{\theta_1^4(z)},$$
$$\zeta(2\omega z)=2\eta z+\frac{1}{2\omega}\frac{\theta_1'(z)}{\theta_1(z)}, \qquad \sigma(2\omega z)=\frac{2\omega}{\pi\theta_1'}\theta_1(z)\mathrm{e}^{2\eta\omega z^2}.$$

Miscellaneous identities and equations:

$$\theta_1=0,\ \theta_1'=\pi\theta_2\theta_3\theta_4,\ \theta_3^4=\theta_2^4+\theta_4^4, \qquad \theta_1(z)\theta_2(z)\theta_3(z)\theta_4(z)=\frac{1}{2}\theta_1(2z)\theta_2\theta_3\theta_4, \qquad 4\mathrm{i}\pi\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial z^2}.$$

§8. Gamma function

8.1. Gamma function

Gamma function generalizes factorial and can be defined by the integrals

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1}\,\mathrm{d}t,$$

$$\Gamma(z) = -\frac{1}{2\mathrm{i}\sin\pi z}\oint_C \mathrm{e}^{-t}(-t)^{z-1}\,\mathrm{d}t, \quad C = \{+\infty + \mathrm{i}\cdot 0,\ 0,\ +\infty - \mathrm{i}\cdot 0\}.$$

The function $1/\Gamma(z)$ is entire and the gamma function has only simple poles in points $-n$ with residue $(-1)^n/n!$.

The characteristic equation for the gamma function is

$$\Gamma(z+1) = z\Gamma(z)$$

providing additionally the logarithmic convexity $\Gamma\Gamma'' > \Gamma'^2$. Other properties include

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \qquad \Gamma(\overline{z}) = \overline{\Gamma(z)},$$

$$\Gamma(nz) = (2\pi)^{\frac{1-n}{2}}\, n^{nz-\frac{1}{2}} \prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right), \quad \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

Particular values

$$\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(1/2) = \sqrt{\pi}, \quad \Gamma(-1/2) = -2\sqrt{\pi},$$

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n-1)!!}{2^n} \equiv \frac{\sqrt{\pi}(2n)!}{4^n n!}.$$

Usually gamma functions enter formulas in combinations. The first is **Pochhammer** symbol:

$$(x)_\nu = \frac{\Gamma(x+\nu)}{\Gamma(x)},$$

in particular,

$$(x)_n = x(x+1)\dots(x+n-1),$$

so that $(x)_0 = 1$ and $(x)_n = 0$ for $x \in \{0, -1, \dots, 1-n\}$. Another combination is **binomial**:

$$\binom{\nu}{\mu} = \frac{\Gamma(\nu+1)}{\Gamma(\mu+1)\Gamma(\nu-\mu+1)} \equiv \frac{\nu!}{\mu!(\nu-\mu)!},$$

Series:

$$\ln\Gamma(1+z) = -\gamma z + \sum_{n=2}^\infty \frac{(-1)^n\zeta(n)}{n} z^n.$$

Infinite product:

$$\frac{1}{\Gamma(z)} = z\mathrm{e}^{\gamma z}\prod_{n=1}^\infty\left(1+\frac{z}{n}\right)\mathrm{e}^{-z/n}.$$

Asymptotic expansions:

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{\mathrm{e}}\right)^z R(z), \qquad \ln R(z) = \sum_{n=1}^\infty \frac{B_{2n}}{2n(2n-1)}\frac{1}{z^{2n-1}} = \frac{1}{12z}\left(1-\frac{1}{30z^2}+\dots\right),$$

$$(x)_\nu = x^\nu\left(1+\frac{\nu(\nu-1)}{2x}+\dots\right),$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \mathrm{e}^{\frac{1}{12n+3/4}}, \text{ relative error } \lesssim 0.6\% \text{ for } n > 0.06,$$

$$\binom{n}{m} = \frac{n^m}{m!}\exp\left[-\frac{m(m-1)}{2n}+O\left(n^{-2}\right)\right], \ n \gg m,$$

$$\binom{n}{m} \geqslant \frac{1}{\sqrt{2\pi n}}\exp\left[-n\left(\mu\ln\mu + (1-\mu)\ln(1-\mu)\right) - \frac{1}{2}\ln\mu(1-\mu) - \frac{1-\mu+\mu^2}{12n\mu(1-\mu)} + O\left(n^{-2}\right)\right], \ m = \mu n, \ n \gg 1$$

(upper limit can be obtained by removing $O(n^{-1})$ term, both approximations are accurate for $m \geqslant 1$),

$$\text{if } \frac{m}{n} = \frac{1\pm\alpha}{2} \text{ then } \binom{n}{m} \approx \frac{2^{n+1}}{\sqrt{2\pi n}}\exp\left[\frac{n}{2}\alpha\ln\frac{1-\alpha}{1+\alpha} - \frac{n+1}{2}\ln\left(1-\alpha^2\right) - \frac{3+\alpha^2}{12n(1-\alpha^2)}\right].$$

8.2. Incomplete gamma function

Incomplete gamma function

$$\gamma(z,x)=\int_0^x \mathrm{e}^{-t}t^{z-1}\,\mathrm{d}t, \qquad \Gamma(z,x)=\int_x^\infty \mathrm{e}^{-t}t^{z-1}\,\mathrm{d}t\equiv\Gamma(z)-\gamma(z,x).$$

It can be expressed via degenerate hypergeometric function as follows:

$$\gamma(z,x)=z^{-1}x^z\mathrm{e}^{-x}\Phi(1,1+z;x), \qquad \Gamma(z,x)=\mathrm{e}^{-x}\Psi(1-z,1-z;x).$$

Note that $\gamma^*=\gamma(z,x)x^{-z}/\Gamma(z)$ is entire function of both arguments z and x.

Series expansion in x:

$$\gamma(z,x)=\sum_{n=0}^\infty \frac{(-1)^n x^{z+n}}{n!(z+n)}.$$

Integer z:

$$\Gamma(n+1,x)=n!\ \mathrm{e}^{-x}\sum_{k=0}^n \frac{x^k}{k!}, \qquad \Gamma(-n,x)=\frac{(-1)^{n+1}}{n!}\left(\mathrm{Ei}(-x)+\mathrm{e}^{-x}\sum_{k=0}^{n-1}\frac{(-1)^k k!}{x^{k+1}}\right).$$

Asymptotic expansion as $x\to+\infty$:

$$\Gamma(1-s,x)=\mathrm{e}^{-x}\sum_{n=0}^\infty \frac{(-1)^n (s)_n}{x^{n+s}}.$$

Miscellaneous identities:

$$\gamma(z+1,x)=z\gamma(z,x)-x^z\mathrm{e}^{-x},$$
$$\Gamma(z,x)=x^{z-1}\mathrm{e}^{-x}\int_0^\infty \mathrm{e}^{-t}\left(1+\frac{t}{x}\right)^{z-1}\mathrm{d}t\equiv\frac{x^z\mathrm{e}^{-x}}{\Gamma(1-z)}\int_0^\infty\frac{\mathrm{e}^{-t}t^{-z}}{x+t}\,\mathrm{d}t,$$
$$x\gamma^*_{xx}+(z+1+x)\gamma^*_x+z\gamma^*=0.$$

8.3. Polygamma functions

Digamma function is the logarithmic derivative of the gamma function $\psi=(\ln\Gamma)'$. **Polygamma** functions are its derivatives $\psi^{(n)}$ so that $\psi^{(0)}\equiv\psi$. Some basic properties and particular values:

$$\psi(\overline{z})=\overline{\psi(z)}, \qquad \psi(z+1)=\psi(z)+\frac{1}{z}, \qquad \psi(1-z)=\psi(z)+\pi\cot\pi z, \qquad \psi(nz)=\frac{1}{n}\sum_{k=0}^{n-1}\psi\left(z+\frac{k}{n}\right)+\ln n,$$
$$\psi(1)=-\gamma, \quad \psi(1/2)=-\gamma-2\ln 2, \quad \psi(1/3)=-\gamma-\frac{\pi}{2\sqrt{3}}-\frac{3\ln 3}{2}, \quad \psi(1/4)=-\gamma-\frac{\pi}{2}-3\ln 2.$$

Series expansion:

$$\psi(1+z)=-\gamma+\sum_{n=2}^\infty(-1)^n\zeta(n)z^{n-1}.$$

Asymptotic expansion as $z\to+\infty$:

$$\psi(z)=\ln z-\frac{1}{2z}-\sum_{n=1}^\infty\frac{B_{2n}}{2n}\frac{1}{z^{2n}}\approx\ln z-\frac{1}{2z}-\frac{1}{12z^2}+\frac{1}{120z^4}.$$

Miscellaneous relations:

$$\psi(z)=-\gamma-\sum_{k=0}^\infty\left(\frac{1}{z+k}-\frac{1}{z+1}\right)=\int_0^\infty\left(\frac{\mathrm{e}^{-t}}{t}-\frac{\mathrm{e}^{-zt}}{1-\mathrm{e}^{-t}}\right)\mathrm{d}t,$$
$$\psi^{(n)}(z)=(-1)^{n+1}n!\sum_{k=0}^\infty\frac{1}{(z+k)^{n+1}}=(-1)^{n+1}\int_0^\infty\frac{\mathrm{e}^{-zt}t^n}{1-\mathrm{e}^{-t}}\,\mathrm{d}t=(-1)^{n+1}n!\ \zeta(n+1,z),$$
$$\sum_{k=1}^n\psi(z+k)=(z+n)\psi(z+n)-z\psi(z)-n.$$

§9. Bessel functions

Bessel functions of the ν-th order of the first and second kind $J_\nu(x)$ and $Y_\nu(x)$ (also known as Neumann function) can be defined as real solutions of the differential equation $x^2y'' + xy' + (x^2 - \nu^2)y = 0$. Complex conjugated (for real arguments) solutions $H^\pm_\nu(x) = J_\nu(x) \pm iY_\nu(x)$ are the **Hankel** functions ($H^{(1)}_\nu \equiv H^+_\nu$ and $H^{(2)}_\nu \equiv H^-_\nu$). **Modified Bessel** functions (Bessel functions of an imaginary argument) of the ν-th order of the first and second kind $I_\nu(x)$ and $K_\nu(x)$ can be defined as real solutions of the differential equation $x^2y'' + xy' - (x^2 + \nu^2)y = 0$.

In what follows Z_ν and C_ν denote one of cylindrical functions J, Y or H. Subscript n implies the integer index.

Series at $x = 0$ ($R = \infty$)

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)k!}\left(\frac{x}{2}\right)^{2k+\nu}, \qquad Y_\nu(x) = \frac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu},$$
$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+\nu+1)k!}\left(\frac{x}{2}\right)^{2k+\nu}, \qquad K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\pi\nu},$$

and in the limit $\nu \to n$ the second kind functions tend to

$$Y_n(x) = -\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} - \frac{2}{\pi}\ln\frac{2}{x}\; J_n(x) - \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\psi(k+1)+\psi(k+n+1)}{(k+n)!k!}\left(\frac{x}{2}\right)^{2k+n},$$
$$K_n(x) = \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} + (-1)^n\ln\frac{2}{x}\; I_n(x) + \frac{(-1)^n}{2}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+n+1)}{(k+n)!k!}\left(\frac{x}{2}\right)^{2k+n}.$$

Relations between Bessel and modified Bessel functions:

$$I_\nu(z) = \mathrm{i}^{-\nu}J_\nu(\mathrm{i}z), \quad K_\nu(z) = \frac{\pi}{2}\mathrm{i}^{\nu+1}H^{(1)}_\nu(\mathrm{i}z), \qquad -\pi < \arg z \leqslant \frac{\pi}{2}, \quad \mathrm{i} \equiv \mathrm{e}^{\mathrm{i}\pi/2};$$
$$\mathrm{i}\pi J_\nu(z) = (-\mathrm{i})^\nu K_\nu(-\mathrm{i}z) - \mathrm{i}^\nu K_\nu(\mathrm{i}z), \quad -\pi Y_\nu(z) = (-\mathrm{i})^\nu K_\nu(-\mathrm{i}z) + \mathrm{i}^\nu K_\nu(\mathrm{i}z), \qquad |\arg z| \leqslant \frac{\pi}{2}.$$

Index inversion formulas:

$$H^\pm_{-\nu} = \mathrm{e}^{\pm\mathrm{i}\pi\nu}H^\pm_\nu, \quad K_{-\nu} = K_\nu;$$
$$Z_{-n} = (-1)^nZ_n, \quad I_{-n} = I_n, \quad J_{1/2-n} = (-1)^nY_{n-1/2}, \quad Y_{1/2-n} = (-1)^{n+1}J_{n-1/2}, \qquad n \in \mathbb{Z}.$$

Differentiation formulas:

$$\left(x^{-\nu}Z_\nu\right)' = -x^{-\nu}Z_{\nu+1}, \qquad (x^\nu Z_\nu)' = x^\nu Z_{\nu-1},$$
$$\left(x^{-\nu}I_\nu\right)' = x^{-\nu}I_{\nu+1}, \qquad (x^\nu I_\nu)' = x^\nu I_{\nu-1},$$
$$\left(x^{-\nu}K_\nu\right)' = -x^{-\nu}K_{\nu+1}, \qquad (x^\nu K_\nu)' = -x^\nu K_{\nu-1}$$

and recurrence relations:

$$Z_{\nu-1} + Z_{\nu+1} = \frac{2\nu}{x}Z_\nu, \qquad Z_{\nu-1} - Z_{\nu+1} = 2Z'_\nu,$$
$$I_{\nu-1} - I_{\nu+1} = \frac{2\nu}{x}I_\nu, \qquad I_{\nu-1} + I_{\nu+1} = 2I'_\nu,$$
$$K_{\nu-1} - K_{\nu+1} = -\frac{2\nu}{x}K_\nu, \qquad K_{\nu-1} + K_{\nu+1} = -2K'_\nu.$$

Wronskians:

$$W[J_\nu, J_{-\nu}](x) = \frac{2\sin\pi\nu}{\pi x}, \quad W[J_\nu, Y_\nu](x) = \frac{2}{\pi x}, \quad W[H^+_\nu, H^-_\nu](x) = -\frac{4\mathrm{i}}{\pi x}, \quad W[I_\nu, K_\nu](x) = -\frac{1}{x}.$$

Asymptotic expansion as $x, z \to +\infty$:

$$J_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \left[A_\nu(x) \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) - B_\nu(x) \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right],$$
$$Y_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \left[A_\nu(x) \sin\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right) + B_\nu(x) \cos\left(x - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right],$$
$$A_\nu(x) = 1 - O\left(x^{-2}\right), \quad B_\nu(x) = \frac{4\nu^2 - 1}{8x} + O\left(x^{-3}\right),$$
$$H_\nu^\pm(z) \sim \sqrt{\frac{2}{\pi z}} \exp\left[\pm \mathrm{i}\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)\right],$$
$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}} \mathrm{e}^z A_\nu^-(z) \pm \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\pi\nu} \frac{1}{\sqrt{2\pi z}} \mathrm{e}^{-z} A_\nu^+(z), \quad -\frac{\pi}{2} < \pm \arg z < \frac{3\pi}{2},$$
$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} \mathrm{e}^{-z} A_\nu^+(z), \quad |\arg z| < \frac{3\pi}{2},$$
$$A_\nu^\pm(z) = \left(1 \pm \frac{4\nu^2 - 1}{8z}\left(1 \pm \frac{4\nu^2 - 9}{16z}\left(1 \pm \ldots \pm \frac{4\nu^2 - (2k-1)^2}{8kz}\left(1 \pm \ldots\right)\right)\right)\right).$$

Uniform approximations (uniform on ξ for $|\arg \xi| < \pi/2$):

$$I_\nu(\nu\xi) \sim \sqrt{\frac{1}{2\pi\nu\sqrt{1+\xi^2}}} \mathrm{e}^{\nu\eta} \left(1 + \sum_{n=1}^{\infty} \frac{1}{\nu^n} u_n\left(\frac{1}{\sqrt{1+\xi^2}}\right)\right),$$
$$K_\nu(\nu\xi) \sim \sqrt{\frac{\pi}{2\nu\sqrt{1+\xi^2}}} \mathrm{e}^{-\nu\eta} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{\nu^n} u_n\left(\frac{1}{\sqrt{1+\xi^2}}\right)\right),$$

where

$$\eta = \sqrt{1+\xi^2} + \ln \frac{\xi}{1 + \sqrt{1+\xi^2}},$$
$$\eta \underset{z\to 0}{\sim} \ln \frac{z}{2} + 1 + \left(\frac{z}{2}\right)^2 - \frac{1}{2}\left(\frac{z}{2}\right)^4 + \ldots, \qquad \eta \underset{z\to\infty}{\sim} z - \frac{1}{2z} + \frac{1}{24z^3} - \ldots,$$
$$u_1(t) = \frac{t(3 - 5t^2)}{24}, \quad u_2(t) = \frac{t^2(81 - 462t^2 + 385t^4)}{1152}.$$

Crossing integrals:

$$\int \left[\left(a^2 - b^2\right) x - \frac{\nu^2 - \mu^2}{x}\right] Z_\nu(ax) C_\mu(bx)\, \mathrm{d}x = bx Z_\nu(ax) C_{\mu-1}(bx) - ax Z_{\nu-1}(ax) C_\mu(bx) + (\nu - \mu) Z_\nu(ax) C_\mu(bx),$$
$$\int Z_\nu^2(ax) x\, \mathrm{d}x = \frac{1}{2}\left(x^2 - \frac{\nu^2}{a^2}\right) Z_\nu^2(ax) + \frac{1}{2} x^2 Z_\nu'^2(ax).$$

Integral representations:

$$J_\nu(x) = \frac{1}{\pi} \int_0^\pi \cos(\nu t - x \sin t)\, \mathrm{d}t,$$
$$K_\nu(x) = \frac{1}{2} \int_{-\infty}^{+\infty} \exp(-x \cosh t - \nu t)\, \mathrm{d}t,$$
$$J_\nu(z) = \frac{1}{2\pi\mathrm{i}} \left(\frac{z}{2}\right)^\nu \oint_C t^{-\nu-1} \exp\left(t - \frac{z^2}{4t}\right) \mathrm{d}t, \quad |\arg z| < \pi, \quad C = \{-\infty + \mathrm{i}\cdot 0,\, 0,\, -\infty - \mathrm{i}\cdot 0\},$$
$$H_\nu^\pm(z) = -\frac{1}{\pi} \int_{C_\pm} \exp\left(-\mathrm{i}z \sin t + \mathrm{i}\nu t\right) \mathrm{d}t, \quad C_\pm = \{-\mathrm{i}\cdot\infty,\, 0,\, \mp\pi,\, \mp\pi + \mathrm{i}\cdot\infty\},$$
$$K_\nu(x) = \frac{1}{2} \int_0^\infty \exp\left(-\frac{t}{2x} - \frac{x}{2t}\right) \frac{\mathrm{d}t}{t^{\nu+1}}.$$

Generating function:

$$\mathrm{e}^{\mathrm{i}z \sin t} = \sum_{n\in\mathbb{Z}} J_n(z) \mathrm{e}^{\mathrm{i}nt}.$$

Addition theorems:

$$\mathrm{e}^{\mathrm{i}n\psi}J_n(cz)=\sum_{k\in\mathbb{Z}}J_{n-k}(az)J_k(bz)\mathrm{e}^{\mathrm{i}k\phi},\qquad c^2=a^2+b^2+2ab\cos\phi,\quad \mathrm{e}^{2\mathrm{i}\psi}=\frac{a+b\mathrm{e}^{+\mathrm{i}\phi}}{a+b\mathrm{e}^{-\mathrm{i}\phi}},$$

$$z^{-\nu}Z_\nu(kz)=\Gamma(\nu)\left(\frac{2}{k}\right)^\nu\sum_{l=0}^{\infty}(\nu+l)\ z_1^{-\nu}J_{\nu+l}(kz_1)\ z_2^{-\nu}Z_{\nu+l}(kz_2)\ C_l^\nu(\cos\theta),\qquad z^2=z_1^2+z_2^2-2z_1z_2\cos\theta,\quad |z_1|<|z_2|,$$

$$\sum_{\substack{n_1+\ldots+n_d=n\\ n_i\in\mathbb{Z}^d}}\prod_{i=1}^{d}\mathrm{e}^{\alpha_i n_i}I_{n_i}(x_i)=\mathrm{e}^{\alpha n}I_n(x),\qquad x\cosh\alpha=\sum_{i=1}^{d}x_i\cosh\alpha_i,\quad x\sinh\alpha=\sum_{i=1}^{d}x_i\sinh\alpha_i.$$

Zeros of the Bessel function J_ν are denoted by $j_{\nu,m}$, where $m\geqslant 1$ is zero's number. They are real for real ν. At large ν or m zeros $j_{\nu,m}$ can be approximated by $\nu u(\pi m/\nu)$, where $u(t)$ satisfies the equation $t=\sqrt{u^2-1}-\arccos(1/u)$ and can be approximated by

$$u(t)=1+\frac{1}{2}(3t)^{2/3}+O\left(t^{4/3}\right),\ t\to 0,\qquad u(t)=(t+\pi/2)-\frac{1}{2(t+\pi/2)}+O\left((t+\pi/2)^{-3}\right),\ t\to\infty.$$

For a half-integer index cylindrical functions reduce to elementary:

$$\begin{aligned}
J_{n+1/2}(x)&=\sqrt{\frac{2}{\pi}}x^{n+1/2}\left(-\frac{\mathrm{d}}{x\,\mathrm{d}x}\right)^n\frac{\sin x}{x}, && J_{-n-1/2}=(-1)^{n+1}Y_{n+1/2},\\
Y_{n+1/2}(x)&=-\sqrt{\frac{2}{\pi}}x^{n+1/2}\left(-\frac{\mathrm{d}}{x\,\mathrm{d}x}\right)^n\frac{\cos x}{x}, && Y_{-n-1/2}=(-1)^{n}J_{n+1/2},\\
K_{n+1/2}(x)&=\sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}\frac{1}{(2z)^k}.
\end{aligned}$$

§10. Legendre functions

10.1. Legendre functions

Legendre functions $P_\nu^\mu(z)$ ($P_\nu^0\equiv P_\nu$) and $Q_\nu^\mu(z)$ can be defined as solutions of the differential equation

$$(1-z^2)w''-2zw'+\left(\nu(\nu+1)-\frac{\mu^2}{1-z^2}\right)w=0. \tag{10.1}$$

Its reduced form is

$$-\frac{\mathrm{d}^2y}{\mathrm{d}\xi^2}+\frac{\mu^2-\frac{1}{4}}{\sin^2\xi}y=\left(\nu+\frac{1}{2}\right)^2y,\qquad \text{where } z=\cos\xi,\ w(z)=\frac{y(\xi)}{\sqrt{\sin\xi}}.$$

The wronskian is

$$W[P_\nu^\mu,Q_\nu^\mu](z)=\frac{4^\mu\mathrm{e}^{\mathrm{i}\mu\pi}}{(1-z^2)}\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)\Gamma\left(\frac{\nu+\mu+2}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu+2}{2}\right)}.$$

We choose the default branch cut to be $(-\infty,-1)$ and $(1,+\infty)$. This is the optimal choice for an argument lying in $(-1,1)$ interval. The imaginary parts of P_ν^μ and Q_ν^μ change their signs on the cut. However sometimes we assume the branch cut to be $(-\infty,1)$. This choice is declared either explicitly or implicitly in formulas containing subexpressions which are real for an argument $z>1$ and complex for $z<1$. In this case the imaginary part of P_ν^μ changes its sign on the cut and Q_ν^μ discontinues in more complicated way. Maple's default choise is $(-\infty,1)$, therefore don't forget to set properly the corresponding environment variable.

Representation via hypergeometric function:

$$P_\nu^\mu(z)=\frac{1}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\frac{\mu}{2}}F\left(-\nu,\nu+1;1-\mu;\frac{1-z}{2}\right),$$

$$Q_\nu^\mu(z)=\frac{\sqrt{\pi}\mathrm{e}^{\mathrm{i}\mu\pi}}{2^{\nu+1}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+3/2)}\frac{(1-z^2)^{\frac{\mu}{2}}}{z^{\nu+\mu+1}}F\left(\frac{\nu+\mu+1}{2},\frac{\nu+\mu+2}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right).$$

Legendre functions with negative indices

$$P^{\mu}_{-\nu-1} = P^{\mu}_{\nu}, \qquad P^{-\mu}_{\nu} = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left(P^{\mu}_{\nu}\cos\pi\mu - \frac{2}{\pi}Q^{\mu}_{\nu}\sin\pi\mu\right), \tag{10.2}$$

$$Q^{\mu}_{-\nu-1} = \frac{Q^{\mu}_{\nu}\sin\pi(\nu+\mu) - \pi P^{\mu}_{\nu}\cos\pi\nu\cos\pi\mu}{\sin\pi(\nu-\mu)}, \qquad Q^{-\mu}_{\nu} = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left(Q^{\mu}_{\nu}\cos\pi\mu + \frac{\pi}{2}P^{\mu}_{\nu}\sin\pi\mu\right),$$

and negative arguments

$$\begin{aligned} P^{\mu}_{\nu}(-z) &= P^{\mu}_{\nu}(z)\cos\pi(\nu+\mu) - \frac{2}{\pi}Q^{\mu}_{\nu}(z)\sin\pi(\nu+\mu), \\ Q^{\mu}_{\nu}(-z) &= -Q^{\mu}_{\nu}(z)\cos\pi(\nu+\mu) - \frac{\pi}{2}P^{\mu}_{\nu}(z)\sin\pi(\nu+\mu), \end{aligned}$$

are another solutions of the Legendre differential equation. Recurrence relations:

$$\begin{aligned} (z^2-1)\frac{\mathrm{d}}{\mathrm{d}z}P^{\mu}_{\nu} &= (\nu-\mu+1)P^{\mu}_{\nu+1} - (\nu+1)zP^{\mu}_{\nu} = \mu z P^{\mu}_{\nu} + \sqrt{1-z^2}P^{\mu+1}_{\nu}, \\ (2\nu+1)zP^{\mu}_{\nu} &= (\nu-\mu+1)P^{\mu}_{\nu+1} + (\nu+\mu)P^{\mu}_{\nu-1}, \end{aligned}$$

and the same formulas for Q^{μ}_{ν}. Special index relations (here the branch cut is $(-\infty,1)$):

$$P^{-\nu-1/2}_{-\mu-1/2}\left(\frac{z}{\sqrt{z^2-1}}\right) = \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\mathrm{i}\mu\pi}}{\Gamma(\nu+\mu+1)}\left(z^2-1\right)^{1/4}Q^{\mu}_{\nu}(z).$$

Products of Legendre functions can be integrated by using the formula

$$\int\left[(\nu-\nu')(\nu+\nu'+1) + \frac{\mu^2-\mu'^2}{1-x^2}\right]w^{\mu}_{\nu}w^{\mu'}_{\nu'} = (1-x^2)\left(w^{\mu}_{\nu}\frac{\mathrm{d}w^{\mu'}_{\nu'}}{\mathrm{d}x} - \frac{\mathrm{d}w^{\mu}_{\nu}}{\mathrm{d}x}w^{\mu'}_{\nu'}\right).$$

For any given $\mu \geq 0$ the Sturm–Liouville problem (see also (12.1))

$$\left\| \begin{array}{l} (1-x^2)y'' - 2xy' + \left(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right)y = 0, \\ y(x) = O(1),\ x\to\pm1 \end{array} \right. \tag{10.3}$$

has the eigenvalues $\nu = n+\mu$, $n\in\mathbb{Z}_+$, corresponding to the eigenfunctions $P^{-\mu}_{n+\mu}(x) \sim (1-x^2)^{\mu/2}\tilde{P}^{0,\mu}_{n}(x)$ with the normalization

$$\int_{-1}^{1}P^{-\mu}_{n+\mu}(x)P^{-\mu}_{k+\mu}(x)\,\mathrm{d}x = \delta_{nk}\frac{n!}{(\mu+n+1/2)\Gamma(2\mu+n+1)}.$$

An alternative spectral problem can be formulated when $\nu > 0$ is fixed in (10.3). In this case the discrete spectrum consists of the eigenvalues $\mu = \nu - n$, $n = \overline{0,[\nu]}$ corresponding to the eigenfunctions $P^{n-\nu}_{\nu}(x)$ with the normalization

$$\int_{-1}^{1}P^{n-\nu}_{\nu}(x)P^{k-\nu}_{\nu}(x)\frac{\mathrm{d}x}{1-x^2} = \delta_{nk}\frac{n!}{(\nu-n)\Gamma(2\nu-n+1)}.$$

The domain $\mu^2 \leq 0$ constitute the continuous spectrum.

Integral representations:

$$P^{-\mu}_{\nu}(\cosh\alpha) = \frac{\sinh^{\mu}\alpha}{2^{\mu}\sqrt{\pi}\Gamma(\mu+1/2)}\int_{-1}^{1}\frac{\left(1-\xi^2\right)^{\mu-1/2}\mathrm{d}\xi}{(\cosh\alpha+\xi\sinh\alpha)^{\mu-\nu}} \equiv \frac{\sinh^{\mu}\alpha}{2^{\mu}\sqrt{\pi}\Gamma(\mu+1/2)}\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\cosh^{\mu+\nu+1}x\ \cosh^{\mu-\nu}(x+\alpha)}.$$

10.2. Legendre polynomials and their dual functions

In case of integer $\mu = m$ the eigenfunctions of (10.3) can be written as P^{m}_{n} (see 10.2) and are called **Legendre quasipolynomials**; functions $P^{0}_{n} \equiv P_n$ are called **Legendre polynomials**. Note that in physical applications the functions $\tilde{P}^{m}_{n} = (-1)^m P^{m}_{n}$ are called Legendre quasipolynomials. The second solution of (10.3) with $\mu = 0$ is given by

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - \sum_{k=1}^{n}\frac{1}{k}P_{k-1}(x)P_{n-k}(x)$$

(for alternative branch cut just change the sign under the logarithm).

Legendre polynomials P_n can be defined also as orthogonal polynomials of an order n on $[-1,1]$.

Some identities[1]:

$$(n+1)P_{n+1} = (2n+1)xP_n - nP_{n-1}, \qquad nP_n = nxP_{n-1} + (x^2-1)P'_{n-1}, \qquad (x^2-1)P'_n = nxP_n - nP_{n-1},$$

$$\int_{-1}^{x}P_n(y)\,\mathrm{d}y = \frac{P_{n+1}(x)-P_{n-1}(x)}{(2n+1)} \equiv \frac{(x^2-1)P'_n}{n(n+1)}, \qquad -\int_{x}^{\infty}Q_n(y)\,\mathrm{d}y = \frac{Q_{n+1}(x)-Q_{n-1}(x)}{(2n+1)}.$$

[1]All identities for P_n in this subsection are valid for Q_n unless otherwise stipulated.

10.3. Hyperspherical harmonics and quasipolynomials

Hyperspherical quasipolynomials $\tilde P_n^{m,a}$ are the eigenfunctions of the following Sturm–Liouville problem ($a > -1/2$ and integer m are parameters):

$$(1-x^2)y'' - 2(a+1)xy' + \left(n(n+2a+1) - \frac{m(m+2a)}{1-x^2}\right) y = 0$$

or

$$\frac{1}{\sin^{2a+1}\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^{2a+1}\theta\frac{\mathrm{d}y}{\mathrm{d}\theta}\right) + \left(n(n+2a+1) - \frac{m(m+2a)}{\sin^2\theta}\right) y = 0$$

with the boundary condition (10.3), which correspond to the eigenvalues $n = m, \ldots, \infty$. Note that $\tilde P_n^{m,a}(\cos\theta)$ is a polynomial in $\cos\theta$ and $\sin\theta$.

This is not a new mathematical function, since the above differential equations are equivalent to (10.3), so that

$$\tilde P_n^{m,a}(x) = \frac{2^a\Gamma(a+1)(2a+1)_{n+m}}{(n-m)!}\left(1-x^2\right)^{-a/2} P_{n+a}^{-m-a}(x).$$

Hyperspherical quasipolynomials are normalized in such a way that $\tilde P_n^{m,0} \equiv \tilde P_n^m$ are **Legendre quasipolynomials** and $\tilde P_n^{0,a} \equiv C_n^{a+1/2}$ are hyperspherical or **Gegenbauer** polynomials. The latter can be defined by the generating function ($|t| < 1$)

$$\left(1-2tx+t^2\right)^{-a-1/2} = \sum_{n=0}^{\infty} \tilde P_n^{0,a}(x)t^n.$$

Explicit expression and normalization:

$$\tilde P_n^{m,a}(x) = \left(1-x^2\right)^{m/2}\frac{\mathrm{d}^m\tilde P_n^{0,a}(x)}{\mathrm{d}x^m}, \qquad \tilde P_n^{0,a}(x) = \frac{(2a+1)_n}{(a+1)_n}\frac{1}{2^n n!}\left(x^2-1\right)^{-a}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^{n+a},$$

$$\int_{-1}^{1}\tilde P_n^{m,a}(x)\tilde P_k^{m,a}(x)\left(1-x^2\right)^a\mathrm{d}x \equiv \int_0^{\pi}\tilde P_n^{m,a}(\cos\theta)\tilde P_k^{m,a}(\cos\theta)\sin^{2a+1}\theta\,\mathrm{d}\theta = \delta_{nk}\frac{2^{1-2a}\pi}{\Gamma\left(a+\frac12\right)^2}\frac{\Gamma(2a+1+n+m)}{(n-m)!(2a+1+2n)},$$

in particular:

$$\tilde P_0^{0,a} = 1, \quad \tilde P_1^{0,a} = (2a+1)x, \quad \tilde P_2^{0,a} = \frac{2a+1}{2}((2a+3)x^2-1), \quad \tilde P_4^{0,a} = \frac{(2a+1)(2a+3)}{6}((2a+5)x^3-3x),$$

$$\tilde P_1^{1,a} = (2a+1)\sin\theta, \quad \tilde P_2^{1,a} = (2a+1)(2a+3)\sin\theta\cos\theta, \qquad \tilde P_2^{2,a} = (2a+1)(2a+3)\sin^2\theta,$$

the case $a = 0$:

$$P_0 = 1,\ P_1 = x,\ P_2 = \frac12(3x^2-1),\ P_3 = \frac12(5x^3-3x),\ P_4 = \frac18(35x^4-30x^2+3),$$

$$\tilde P_1^1 = \sin\theta, \qquad \tilde P_2^1 = 3\sin\theta\cos\theta,\ \tilde P_2^2 = 3\sin^2\theta, \qquad \tilde P_3^1 = \frac32(5\cos^2\theta-1)\sin\theta,\ \tilde P_3^2 = 15\sin^2\theta\cos^2\theta,\ \tilde P_3^3 = 15\sin^3\theta.$$

Hyperspherical harmonics $Y_{l_n\ldots l_1 m}(\theta_n,\ldots,\theta_1,\phi) \equiv Y_\lambda(\boldsymbol{n})$ are the eigenfunctions of the Laplace operator Δ_n on a hypersphere in $\mathbb{R}^{n+2}$, here

$$\Delta_n = \frac{1}{\sin^n\theta_n}\frac{\partial}{\partial\theta_n}\left(\sin^n\theta_n\frac{\partial}{\partial\theta_n}\right) + \frac{\Delta_{n-1}}{\sin^2\theta_n}, \qquad \Delta_0 = \frac{\partial^2}{\partial\phi^2}, \qquad \theta_k \in (0,\pi),\ \phi \in (0,2\pi).$$

The eigenvalues are $\Delta_n Y = -l_n(l_n+n)Y$, where $l_n \geqslant \ldots \geqslant l_1 \geqslant 0$ and $-l_1 \leqslant m \leqslant l_1$, so that the multiplicity is $\frac{2l_n+n}{l_n+n}\binom{l_n+n}{n}$. The principal hyperspherical harmonic is $Y_{l_n0\ldots0}(\theta_n,\ldots) \equiv Y_{l_n}(\theta_n)$ depends only on θ_n, by its rotation all other Y_λ of the order l_n can be obtained.

Explicit expression and normalization:

$$Y_\lambda(\boldsymbol{n}) = \Phi_m(\phi)\prod_{k=1}^{n}\tilde P_{l_k}^{l_{k-1},\frac{k-1}{2}}(\theta_k), \quad (l_0 \equiv |m|), \tag{10.4}$$

$$\|Y_\lambda\|^2 = \|\Phi_m\|^2\prod_{k=1}^{n}\frac{2^{2-k}\pi}{\Gamma(k/2)^2}\frac{(l_k+l_{k-1}+k-1)!}{(l_k-l_{k-1})!(2l_k+k)},$$

where Φ_m can be chosen in exponential form, $\mathrm{e}^{\mathrm{i}m\phi}$, $\|\Phi_m\|^2 = 2\pi$, or trigonometric,

$$\Phi_m(\phi) = \begin{cases}\cos m\phi, & m \geqslant 0,\\ \sin|m|\phi, & m < 0,\end{cases} \qquad \|\Phi_m\|^2 = \begin{cases}\pi, & m \neq 0,\\ 2\pi, & m = 0.\end{cases}$$

Addition theorem:

$$\sum_{l_{n-1}\ldots l_1 m}\frac{Y_\lambda(\boldsymbol{n})\overline{Y_\lambda(\boldsymbol{n}')}}{\|Y_\lambda\|^2} = \sqrt{\frac{(2l+n)}{(4\pi)^{\frac{n+1}{2}}}\frac{(l+n-1)!}{l!}}\ \frac{Y_{l_n}(\theta)}{\|Y_{l_n}\|},$$

where θ is the angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$.

10.4. Spherical harmonics

In quantum mechanics **spherical harmonics** are chosen in the following form (Wikipedia, Mathematica)

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\,\tilde{P}_l^{|m|}(\cos\theta)\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{10.5}$$

so that $Y_{l,-m} = (-1)^m\overline{Y_{lm}}$. This definition differs from (10.4). The lowest order spherical harmonics:

$$Y_{00} = \sqrt{\frac{1}{4\pi}},\quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta,\quad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\mathrm{e}^{\pm\mathrm{i}\phi},\quad Y_{20} = \sqrt{\frac{5}{16\pi}}(1-3\cos^2\theta),$$
$$Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\mathrm{e}^{\pm\mathrm{i}\phi},\quad Y_{2\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\mathrm{e}^{\pm\mathrm{i}2\phi},\quad Y_{30} = \sqrt{\frac{7}{16\pi}}\cos\theta(5\cos^2\theta-3),$$
$$Y_{3\pm1} = \mp\sqrt{\frac{21}{64\pi}}\sin\theta(5\cos^2\theta-1)\mathrm{e}^{\pm\mathrm{i}\phi},\quad Y_{3\pm2} = \sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta\mathrm{e}^{\pm\mathrm{i}2\phi},\quad Y_{3\pm3} = \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta\mathrm{e}^{\pm\mathrm{i}3\phi}.$$

The real valued (tesseral) spherical harmonics are

$$\tilde{Y}_{l0} = Y_{l0},\quad \tilde{Y}_{lm} = (-1)^m\sqrt{2}\Re Y_{l|m|}\ \text{for}\ m>0,\quad \tilde{Y}_{lm} = (-1)^m\sqrt{2}\Im Y_{l|m|}\ \text{for}\ m<0. \tag{10.6}$$

For a fixed l this basis is usually ordered as follows: $\{0,1,-1,2,-2,\ldots\}$.

If the coordinate system is rotated by Euler's angles in y-convention so that $\boldsymbol{r}' = R(\alpha,\beta,\gamma)\boldsymbol{r}$, the spherical harmonics transform by the formula

$$Y_{lm}(\theta',\phi') = \sum_{m'=-l}^{l} D^l_{mm'}(\alpha,\beta,\gamma)Y_{lm'}(\theta,\phi),$$

where D is unitary Wigner's rotation operator given by

$$D^l_{mm'}(\alpha,\beta,\gamma) = \mathrm{e}^{\mathrm{i}m\alpha+\mathrm{i}m'\gamma}\,d^l_{mm'}(\beta),\qquad \overline{D^l_{mm'}}(\alpha,\beta,\gamma) = D^l_{m'm}(-\gamma,-\beta,-\alpha),$$
$$d^l_{mm'}(\beta) = \sqrt{(l+m)!(l-m)!(l+m')!(l-m')!}\sum_{k=0}^{2l}\frac{(-1)^{l-k-m'}\left(\cos\frac{\beta}{2}\right)^{2k+m+m'}\left(\sin\frac{\beta}{2}\right)^{2(l-k)-m-m'}}{k!(k+m+m')!(l-k-m)!(l-k-m')!}$$

(in the last sum all terms with a negative argument of the factorial are zero). The matix $d_{mm'}$ is antisymmetric wrt to the main diagonal and symmetric wrt to the antidiagonal:

$$d^l_{m'm} = d^l_{-m,-m'} = (-1)^{m+m'}d^l_{mm'}.$$

In particular,

$$D^l_{0m}(\alpha,\beta,\gamma) = (-1)^m\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\beta,\gamma).$$

The decomposition

$$Y_{l_1m_1}Y_{l_2m_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2}(-1)^{m_1+m_2}\sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}}\begin{pmatrix} l_1 & l_2 & l\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l\\ m_1 & m_2 & -m_1-m_2\end{pmatrix}Y_{l,m_1+m_2},$$

where in braces are Wigner 3j-symbols, can be used for evaluation of matrix elements

$$\langle l_1m_1\,|f(\theta,\phi)|\,l_2m_2\rangle \equiv \int \overline{Y_{l_1m_1}}\ f\ Y_{l_2m_2}\,\mathrm{d}\Omega.$$

In particular, the selection rules for $\langle l_1m_1\,|lm|\,l_2m_2\rangle$ matrix elements are: $m = m_1 - m_2$, $|l_1-l_2| \leqslant l \leqslant l_1+l_2$, and l_1+l_2+l must be even. For $l=1$ nonzero elements are the following ones:

$$\langle l-1,m|\cos\theta|l,m\rangle = \langle l,m|\cos\theta|l-1,m\rangle = \sqrt{\frac{l^2-m^2}{4l^2-1}},$$
$$\langle l,m|\sin\theta\ \mathrm{e}^{\mathrm{i}\phi}|l-1,m-1\rangle = \langle l-1,m-1|\sin\theta\ \mathrm{e}^{-\mathrm{i}\phi}|l,m\rangle = \sqrt{\frac{(l+m)(l+m-1)}{4l^2-1}},$$
$$\langle l-1,m|\sin\theta\ \mathrm{e}^{\mathrm{i}\phi}|l,m-1\rangle = \langle l,m-1|\sin\theta\ \mathrm{e}^{-\mathrm{i}\phi}|l-1,m\rangle = -\sqrt{\frac{(l-m)(l-m+1)}{4l^2-1}}.$$

Other integrals:

$$\iint \overline{Y_{lm}}(\boldsymbol{n}) Y_{l'm'}(\boldsymbol{n'}) f(\boldsymbol{nn'})\, \mathrm{d}\Omega\, \mathrm{d}\Omega' = \delta_{ll'}\delta_{mm'}\ 2\pi \int_{-1}^{1} P_l(x) f(x)\, \mathrm{d}x.$$

The basis for **harmonic polynomials** consists of the polynomials

$$h_{lm}(\boldsymbol{r}) = \sqrt{\frac{4\pi}{2l+1}}\ r^l Y_{lm}(\theta, \phi). \tag{10.7}$$

If Y are normalized in (10.7) then the addition theorem reads

$$h_{lm}\left(\boldsymbol{r}_1 + \boldsymbol{r}_2\right) = \sum_{l_1=0}^{\infty} \sum_{m_1=-l_1}^{l_1} \sqrt{\binom{l+m}{l_1+m_1}\binom{l-m}{l_1-m_1}}\ h_{l_1 m_1}\left(\boldsymbol{r}_1\right)\ h_{l-l_1, m-m_1}\left(\boldsymbol{r}_2\right),$$

note that all the terms in the above sum with $|m - m_1| > l - l_1$ are zero.

§11. Hypergeometric functions

Generalized **hypergeometric** function is defined by the series

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!}.$$

We use the brief notations for hypergeometric functions omitting their indexes:

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \equiv F(a_1, \ldots, a_p; b_1, \ldots, b_q; z).$$

11.1. Hypergeometric function 2/1

Hypergeometric function ${}_2F_1$ is defined by the series

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

(the convergence radius is 1 at least). If a or b is a nonpositive integer $-m$ then F is a polynomial of the order m. If $c \in \mathbb{Z}_-$ then F is undefined but

$$\lim_{c \to -m+1} \frac{F(a, b; c; z)}{\Gamma(c)} = \frac{(a)_m (b)_m}{m!} z^m F(a+m, b+m; 1+m; z).$$

An alternative definition is via the hypergeometric differential equation

$$z(1-z)w'' + (c - (a+b+1)z)w' - abw = 0, \tag{11.1}$$

whose reduced form is

$$-y'' + \left(\frac{\lambda^2 - 1}{4z^2} + \frac{\mu^2 - 1}{4(1-z)^2} + \frac{\lambda^2 + \mu^2 - \nu^2 - 1}{4z(1-z)}\right) y = 0,$$

here $w(z) = z^{-c/2}(1-z)^{(c-a-b-1)/2} y(z)$, $\lambda = 1-c$, $\mu = c-a-b$, $\nu = a-b$. The solution of (11.1) for $c \notin \mathbb{Z}_-$ is

$$w_1(z) = F(a, b; c; z), \qquad w_2(z) = z^{1-c} F(a-c+1, b-c+1; 2-c; z)$$

with the wronskian $(1-c)z^{-c}(1-z)^{(c-a-b-1)}$. Under any choice if $w_1(0) = 1$ then $w_2(z)$ diverges at 0 as z^{1-c} for $c \neq 1$ and as $\ln z$ for $c = 1$. If $c = m$ where $m \in \mathbb{N}$ then

$$w_1(z) = F(a, b; m; z), \qquad w_2(z) = F(a, b; m; z) \ln z - \sum_{n=1}^{m} \frac{(n-1)!(1-m)_n}{(1-a)_n (1-b)_n} z^{-n}$$

$$+ \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(m)_n n!} z^n \left[\psi(a+n) - \psi(a) + \xi(b+n) - \psi(b) - \psi(m+n) + \psi(m) - \psi(n+1) + \xi(1)\right].$$

If $c<1$ then the substitution $w(z)=z^{1-c}\tilde{w}(z)$ into (11.1) yields the hypergeometric equation with $\tilde{a}=a-c+1$, $\tilde{b}=b-c+1$, $\tilde{c}=2-c$.

Transformation formulas:

$$\begin{aligned}F(a,b;c;z)&=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-z)+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z)\\&=(-1)^a\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}z^{-a}F\left(a,a-c+1;a-b+1;\frac{1}{z}\right)+(-1)^b\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}z^{-b}F\left(b,b-c+1;b-a+1;\frac{1}{z}\right)\\&=(1-z)^{-a}F\left(a,c-b;c;\frac{z}{z-1}\right)\\&=(1-z)^{c-a-b}F(c-a,c-b;c,z).\end{aligned}$$

Integral representation:

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t.$$

11.2. Hypergeometric function 1/1

Degenerate hypergeometric function (the **first Kummer** function) ${}_1F_1(a;c;z)\equiv M(a,c,z)$ can be defined as the solution of the degenerate hypergeometric equation

$$zw''+(c-z)w'-aw=0. \tag{11.2}$$

The second solution gives the **second Kummer** function

$$U(a,c,z)=\frac{\pi}{\sin\pi c}\left[\frac{M(a,c,z)}{\Gamma(1+a-c)\Gamma(c)}-z^{1-c}\frac{M(1+a-c,2-c,z)}{\Gamma(a)\Gamma(2-c)}\right]\equiv z^{-a}{}_2F_0(a,1+a-c;-1/z).$$

Note that sometimes M and U are denoted by Φ and Ψ. The function M is entire whereas the function U has the logarithmic branch cut $(-\infty,0)$. Series at zero gives an alternative definition:

$$M(a,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}\frac{z^n}{n!},\qquad |z|<1.$$

Integer parameters requires a separate consideration. Let m is a nonnegative integer. If $a=-m$ then M is a polynomial of the order m and U becomes proportional to M, in this case the function

$$\lim_{a\to-m}\Gamma(a)U(a,c,z)\equiv-\frac{\pi z^{1-c}M(1-m-c,2-c,z)}{\sin\pi c\;\Gamma(2-c)}$$

gives the second solution of (11.2). If $c=m+1$ then U is undefined but the integral representation gives the so called logarithmic solution

$$U(a,m+1,z)\equiv\frac{(-1)^{m-1}}{m!\;\Gamma(a-m)}\left[M(a,m+1,z)\ln z+\sum_{n=0}^{\infty}[\psi(a+n)-\psi(1+n)-\psi(1+m+n)]\,\frac{(a)_n}{(m+1)_n}\frac{z^n}{n!}\right]+\frac{(m-1)!}{\Gamma(a)}\sum_{n=0}^{m-1}\frac{(a-m)_n}{(1-m)_n}\frac{z^{n-m}}{n!}.$$

Finally, if $c=-m$ then M is undefined but

$$\lim_{c\to-m+1}\frac{M(a,c,z)}{\Gamma(c)}=\frac{(a)_m}{m!}z^mM(a+m,1+m,z)$$

and U can be evaluated by the identity

$$U(a,1-m,z)\equiv z^mU(a+m,1+m,z).$$

Integral representations:

$$\begin{aligned}M(a,c,z)&=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \mathrm{e}^{zt}t^{a-1}(1-t)^{c-a-1}\,\mathrm{d}t,\\U(a,c,z)&=\frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-zt}t^{a-1}(1+t)^{c-a-1}\,\mathrm{d}t,\\M(a,c,z)&=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\frac{\Gamma(c)\Gamma(a+s)\Gamma(-s)}{\Gamma(a)\Gamma(c+s)}(-z)^s\,\mathrm{d}s,\quad -\Re a<\sigma<0,\ |\arg(-z)|<\frac{\pi}{2},\\U(a,c,z)&=\frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty}\frac{\Gamma(a+s)\Gamma(1-c-s)\Gamma(-s)}{\Gamma(a)\Gamma(a-c+1)}(-z)^s\,\mathrm{d}s,\quad -\Re a<\sigma<\min(0,1-\Re c),\ |\arg z|<\frac{3\pi}{2}.\end{aligned}$$

Asymptotic expansions as $z \to \infty$:

$$U(a,c,z) \sim z^{-a} \sum_{n=0}^{\infty} \frac{(a)_n (1+a-c)_n}{n!} (-z)^{-n},$$

$$M(a,c,z) \sim \mathrm{e}^{\pm \mathrm{i}\pi a} \frac{\Gamma(c)}{\Gamma(c-a)} U(a,c,z) + \frac{\Gamma(c)}{\Gamma(a)} \mathrm{e}^z z^{a-c} \sum_{n=0}^{\infty} \frac{(c-a)_n (1-a)_n}{n!} z^{-n},$$

where $\pm$ means the upper or lower half-plane.

Whittaker functions $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are the solutions of the differential equation

$$y'' + \left(-\frac{1}{4} + \frac{\mu}{z} + \frac{\frac{1}{4} - \nu^2}{z^2} \right) y = 0.$$

It can be obtained from (11.2) by the transformation $w(z) = \mathrm{e}^{z/2} z^{-c/2} y(z)$ with $\mu = c/2 - a$ and $\nu = c/2 - 1/2$ (note that $a = -n$, $n \in \mathbb{Z}_+$ corresponds to $\mu = \nu + n + 1/2$). In this way

$$\begin{pmatrix} M \\ W \end{pmatrix}_{\mu,\nu} (z) = z^{\nu+1/2} \mathrm{e}^{-z} \begin{pmatrix} M \\ U \end{pmatrix} (1/2 - \mu + \nu, 2\nu + 1, z).$$

Some integrals:

$$\int_0^\infty x^{\alpha-1} M^2_{s+n+1/2,s}(x)\, \mathrm{d}x = n!\, \Gamma(\alpha + 2s + 1 + n) \left(\frac{\Gamma(2s+1)}{\Gamma(2s+1+n)} \right)^2 F(-n, -\alpha, -\alpha; 1, -\alpha - 2s - n; 1),\ n \in \mathbb{Z}_+.$$

§12. Orthogonal polynomials

For Legendre and Gegenbauer polynomials see the spherical harmonics.

12.1. Jacobi polynomials

Jacobi polynomials P_n^{ab} are orthogonal polynomials on $[-1,1]$ with

$$\int_{-1}^1 P_n^{ab}(x) P_m^{ab}(x) (1-x)^a (1+x)^b\, \mathrm{d}x = \delta_{nm} \frac{2^{a+b+1}}{2n+a+b+1} \frac{(n+a)!(n+b)!}{n!(n+a+b)!}.$$

They satisfy the differential equation

$$(1-x^2)y'' + (b - a - (a+b+2)x)y' + n(n+a+b+1)y = 0$$

and therefore reduce to the hypergeometric function:

$$P_n^{ab}(x) = \binom{n+a}{n} F\left(-n; n+a+b+1, a+1; \frac{1-x}{2} \right).$$

Explicit form:

$$P_n^{ab}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-a} (1+x)^{-b} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[(1-x)^{n+a} (1+x)^{n+b} \right] \equiv \left(\frac{x-1}{2} \right)^n \sum_{k=0}^n \binom{n+a}{k} \binom{n+b}{n-k} \left(\frac{x+1}{x-1} \right)^k.$$

Particular values:

$$P_n^{ab}(-x) = (-1)^n P_n^{ab}(x), \qquad P_n^{ab}(1) = \binom{n+a}{n}, \qquad P_0^{ab}(x) = 1, \qquad P_1^{ab}(x) = \frac{a-b}{2} + \left(1 + \frac{a+b}{2}\right) x.$$

Recurrence relations:

$$2(n+1)(n+a+b+1)(2n+a+b)P_{n+1}^{ab}(x) = \left[(2n+a+b+1)(a^2-b^2) + (2n+a+b)_3 x \right] P_n^{ab}(x) - 2(n+a)(n+b)(2n+a+b+2) P_{n-1}^{ab}(x).$$

Generating function:

$$\frac{2^{a+b}}{R(1-t+R)^a (1+t+R)^b} = \sum_{n=0}^\infty P_n^{ab}(x) t^n, \qquad R = \sqrt{1 - 2xt + t^2}.$$

For any given $a, b \geqslant 0$ the Sturm–Liouville problem

$$\left\| \begin{array}{l} (1-x^2)u'' - 2xu' + \left(\nu(\nu+1) - \frac{a^2/2}{1-x} - \frac{b^2/2}{1+x} \right) u = 0, \\ u(x) = O(1),\ x \to \pm 1 \end{array} \right. \tag{12.1}$$

has the eigenvalues $\nu = n + \frac{a+b}{2}$, $n \in \mathbb{Z}_+$, corresponding to the eigenfunctions $(1-x)^{a/2}(1+x)^{b/2} P_n^{ab}(x)$.

12.2. Chebyshev polynomials

Chebyshev polynomials T_n satisfy the differential equation

$$(1-x^2)y'' - xy' + n^2 y = 0,\ x \in [-1,1].$$

Their orthogonalization properties, recurrence relations, explicit form, and particular values:

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,\mathrm{d}x = \delta_{mn} \begin{cases} \pi/2, & m \neq 0, \\ \pi, & m = 0, \end{cases}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

$$T_n(x) = \cos(n \arccos x), \qquad T_0 = 1,\ T_1 = x,\ T_2 = 2x^2 - 1,\ T_3 = 4x^3 - 3x,\ T_4 = 8x^4 - 8x^2 + 1.$$

12.3. Laguerre polynomials

Laguerre polynomials $L_n^a(x)$ ($L_n^0 \equiv L_n$) can be defined as the eigenfunctions of the Sturm–Liouville problem in an infinite domain

$$\left\| \begin{array}{l} xy'' + (a+1-x)y' + ny = 0, \\ y(x) = O(1),\ x \to +0, \\ y(x) = o(\mathrm{e}^{\varepsilon x}),\ x \to +\infty. \end{array} \right. \tag{12.2}$$

Laguerre polynomials L_n^a are orthogonal polynomials of an order n with the following weight:

$$\int_0^\infty L_n^a(x) L_m^a(x) \mathrm{e}^{-x} x^a\,\mathrm{d}x = \delta_{nm} \frac{\Gamma(n+a+1)}{n!}.$$

Note that sometimes alternative definition is used: $L_{\mathrm{alt}}(n,a,x) = (-1)^a n! L_{n-a}^a(x)$. Since (12.2) is the degenerate hypergeometric equation (11.2)

$$L_n^a(x) = \binom{n+a}{n} M(-n; a+1; x).$$

Explicit form:

$$L_n^a = \frac{1}{n!} \mathrm{e}^x x^{-a} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (\mathrm{e}^{-x} x^{n+a}) \equiv \sum_{k=0}^{n} \binom{n+a}{n-k} \frac{(-x)^k}{k!},$$

in particular,

$$L_0^a = 1, \quad L_1^a = a + 1 - x, \quad L_2^a = \frac{(a+1)(a+2)}{2} - (a+2)x + \frac{1}{2}x^2.$$

Recurrence relations in n:

$$(n+1)L_{n+1}^a = (2n+a+1-x)L_n^a - (n+a)L_{n-1}^a, \qquad x\frac{\mathrm{d}}{\mathrm{d}x} L_n^a = nL_n^a - (n+a)L_{n-1}^a,$$

other recurrence relations:

$$L_n^a = L_n^{a-1} + L_{n-1}^a, \qquad xL_n^{a+1} = (n+a)L_{n-1}^a - (n-x)L_n^a, \qquad \frac{\mathrm{d}}{\mathrm{d}x} L_n^a = -L_{n-1}^{a+1}.$$

Generating functions

$$\frac{1}{(1-t)^{a+1}} \exp\frac{-tx}{1-t} = \sum_{n=0}^{\infty} L_n^a(x) t^n,$$

$$\mathrm{e}^{-tx}(1+t)^a = \sum_{n=0}^{\infty} L_n^{a-n}(x) t^n.$$

Uniform approximation for $n \to \infty$ (valid for $0 \leqslant y < 1$):

$$(2y)^{a+1/2} \mathrm{e}^{-\nu y^2} L_n^a\left(2\nu y^2\right) = \left(1-y^2\right)^{-1/4} \sqrt{\xi} J_a(\nu\xi),\ \text{where } \nu = 2n+a+1,\ \xi = y\sqrt{1-y^2} + \arcsin y.$$

12.4. Hermite polynomials

Hermite polynomials H_n satisfy the differential equation

$$y'' - 2xy' + 2ny = 0, \quad x \in \mathbb{R}.$$

Their orthogonalization properties, recurrent relations, explicit form, and particular values:

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) \mathrm{e}^{-x^2}\, \mathrm{d}x = 2^n n! \sqrt{\pi}\, \delta_{mn},$$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad H_n'(x) = 2nH_{n-1}(x),$$

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n \mathrm{e}^{-x^2}}{\mathrm{d}x^n} = \sum_{m=0}^{[n/2]} \frac{(-1)^m n!}{m!(n-2m)!} (2x)^{n-2m},$$

$$H_0 = 1,\ H_1 = 2x,\ H_2 = 4x^2 - 2,\ H_3 = 8x^3 - 12x,\ H_4 = 16x^4 - 48x^2 + 12.$$

Relation to other functions:

$$H_n(x) = 2^n \operatorname{sgn} x\ U(-n/2, 1/2, x^2),$$

$$H_{2k}(x) = (-1)^k 2^{2k} k!\ L_k^{-1/2}\left(x^2\right), \qquad H_{2k+1}(x) = (-1)^k 2^{2k+1} k!\ x L_k^{1/2}\left(x^2\right).$$

Some integrals:

$$\int_0^{\infty} x^{\alpha-1} \mathrm{e}^{-x^2} H_n(x)\, \mathrm{d}x = \begin{cases} (-1)^k 2^{2k} \left(\frac{2-\alpha}{2}\right)_k \Gamma\left(\frac{1+\alpha}{2}\right), & n = 2k+1, \\ (-1)^k 2^{2k-1} \left(\frac{1-\alpha}{2}\right)_k \Gamma\left(\frac{\alpha}{2}\right), & n = 2k, \end{cases}$$

$$\int_0^{\infty} \mathrm{e}^{-\alpha x^2} H_{2k}(x)\, \mathrm{d}x = \frac{\sqrt{\pi}(2k)!}{2k!} \frac{(1-\alpha)^k}{\alpha^{k+1/2}},$$

$$\int_0^{\infty} \mathrm{e}^{-\alpha x^2} x H_{2k+1}(x)\, \mathrm{d}x = \frac{\sqrt{\pi}(2k+1)!}{2k!} \frac{(1-\alpha)^k}{\alpha^{k+1/2}},$$

$$\int_0^{\infty} \mathrm{e}^{-\alpha(x-b)^2} H_n(x)\, \mathrm{d}x = \sqrt{\frac{\pi}{\alpha}} \left(\sqrt{\frac{\alpha-1}{\alpha}}\right)^n H_n\left(b\sqrt{\frac{\alpha}{\alpha-1}}\right).$$

§13. Generalized functions

Heaviside function $\theta(x)$ is defined as the piecewise function

$$\theta(x) \equiv \frac{1 + \operatorname{sgn} x}{2} = \begin{cases} 0, & x < 0, \\ 1/2, & x = 0, \\ 1, & x > 0. \end{cases}$$

It can be expressed via the functional sequences:

$$\theta(x) = \lim_{a\to+\infty} \left(\frac{1}{2} + \frac{1}{\pi} \arctan ax\right) = \lim_{a\to+\infty} \frac{1}{2}\left(1 + \operatorname{erf}(ax)\right) = \lim_{a\to+\infty} 2^{-\exp(-ax)} = \lim_{a\to+\infty} \frac{1}{\pi} \int_{-\infty}^{ax} \frac{\sin y}{y}\, \mathrm{d}y.$$

Its Fourier transformation is determined by the integral

$$\theta(x) = \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\sin kx}{k}\, \mathrm{d}k.$$

Dirac function $\delta(x)$ can be defined as the derivative of the Heaviside function $\delta = \theta'$. Generally the Dirac function and its derivatives are defined by the functionals

$$\int_a^b f(y) \delta^{(n)}(y - x)\, \mathrm{d}y = \begin{cases} \frac{1}{2}(-1)^n (f^{(n)}(x+0) + f^{(n)}(x-0)), & a < x < b, \\ \frac{1}{2}(-1)^n f^{(n)}(a+0), & x = a, \\ \frac{1}{2}(-1)^n f^{(n)}(b-0), & x = b, \\ 0, & \text{otherwise}. \end{cases}$$

It can be expressed via the functional sequences:

$$\delta(x) = \lim_{\varepsilon\to+0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \lim_{\varepsilon\to+0}\frac{1}{\pi}\Im\frac{1}{x-\mathrm{i}\varepsilon} = \lim_{\sigma\to+0}\frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{x^2}{2\sigma^2}} = \lim_{k\to+\infty}\frac{\sin kx}{\pi x}.$$

Its Fourier transformation is determined by the integral

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

If ϕ has only single zeros denoted by ξ then

$$\delta(\phi(x)) = \sum_{\xi}\frac{\delta(x-\xi)}{|\phi'(\xi)|}.$$

The multidimensional Dirac function is defined as follows:

$$\delta(x) = \prod_i \delta(x_i), \qquad \delta(\mathbf{A}x - b) = \frac{1}{\det\mathbf{A}}\delta\left(x - \mathbf{A}^{-1}b\right).$$

Identities:

$$\delta(\boldsymbol{r}-\boldsymbol{a}) = \frac{1}{r^2\sin\theta}\delta(r-a)\delta(\theta-\theta_a)\delta(\phi-\phi_a), \qquad \int f(r)\delta(\boldsymbol{r})\,\mathrm{d}V = \int f(r)\delta(r)\,\mathrm{d}r,$$

$$\delta(\boldsymbol{r}) = \frac{1}{4\pi}\boldsymbol{\nabla}\left(\frac{\boldsymbol{r}}{r^3}\right) = -\frac{1}{4\pi}\Delta\left(\frac{1}{r}\right), \qquad \int_{\psi(\boldsymbol{r})=0} f\,\mathrm{d}S = \int_{\psi(\boldsymbol{r})\geqslant 0} f\ \delta(\psi(\boldsymbol{r}))\,|\boldsymbol{\nabla}\psi|\ \mathrm{d}V.$$

$$|x|' = \operatorname{sgn} x, \qquad (\operatorname{sgn} x)' = 2\delta(x).$$