Handbook on basic analysis

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§1. Constants

$\pi \approx 3.1416, \quad \mathrm{e} \approx 2.7183, \quad \ln 10 \approx 2.3026, \quad \lg 2 \approx 0.30103.$
Euler's constant $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} k^{-1} - \ln n\right) \approx 0.5772.$

§2. Sums and products

2.1. Series expansion

Taylor's formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(y)(x-y)^n \,\mathrm{d}y = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1},\ 0 < \xi < x.$$

Multidimensional Taylor's formula:

$$f(x_1,\ldots,x_d) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{i=1}^{d} x_i \partial_i\right)^n f = \sum_{n_1,\ldots,n_d \geqslant 0} f^{(n_1,\ldots,n_d)} \frac{x_1^{n_1}\ldots x_d^{n_d}}{n_1!\ldots n_d!}.$$

Series expansion of some elementary functions:

$$(1+x)^\mu = \sum_{n=0}^{\infty} \binom{\mu}{n} x^n = 1 + \mu x + \frac{\mu(\mu-1)}{2} x^2 + \ldots, \qquad \frac{1}{(1-x)^{m+1}} = \sum_{n=0}^{\infty} \binom{m+n}{m} x^n.$$

2.2. Finite sums and products

Hypergeometric sums:

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}, \qquad \sum_{k=0}^{n} k x^k = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2},$$

$$\sum_{k=0}^{\infty} q^k (\cos kx + \mathrm{i} \sin kx) = \frac{1 - q\cos x + \mathrm{i} q \sin x}{1 - 2q\cos x + q^2}, \ |q| < 1.$$

Binomial sums:

$$\sum_{k=0}^{n} \binom{n}{k} \binom{a}{b+k} = \binom{a+n}{b+n}.$$

“Number of points” sums:

$$\sum_{1\leqslant k_1 \leqslant k_2 \leqslant \ldots \leqslant k_m \leqslant n} 1 = \binom{n+m-1}{m}, \qquad \sum_{1\leqslant k_1 < k_2 < \ldots < k_m \leqslant n} 1 = \binom{n}{m}.$$

Multinomial coefficients and Pascal’s triangle for them:

$$\left(\sum_{k=1}^{m} x_k\right)^n = n! \sum_{n_k \geqslant 0}^{\sum_k n_k = n} \prod_{k=1}^{m} \frac{x_k^{n_k}}{n_k!},$$

$$\sum_{k=0}^{m} \frac{(n-1)!}{n_1! \ldots n_{k-1}! (n_k - 1)! n_{k+1}! \ldots n_m!} = \frac{n!}{n_1! \ldots n_m!}, \qquad n = n_1 + \ldots + n_m.$$

2.3. Infinite sums and products

Polynomial products:

$$\prod_{n=0}^{\infty} \left(1 + \frac{y^2}{(x+n)^2}\right) = \left|\frac{\Gamma(x)}{\Gamma(x+\mathrm{i}y)}\right|^2, \qquad \prod_{n\in\mathbb{Z}} \frac{(x_1+n)^2 + y_1^2}{(x_2+n)^2 + y_2^2} = \frac{\sin^2 \pi x_1 + \sinh^2 \pi y_1}{\sin^2 \pi x_2 + \sinh^2 \pi y_2}.$$

Products reducible to theta function:

$$\prod_{n=1}^{\infty} \left(1 - q^{2n}\right) = \sqrt[3]{\frac{\theta_1'}{2\sqrt[4]{q}}}, \quad \prod_{n=0}^{\infty} \left(1 - q^{2n+1}\right) = \sqrt[6]{\frac{2\sqrt[4]{q}\theta_4^3}{\theta_1'}}, \quad \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)\left(1 - 2q^{2n}\cos 2x + q^{4n}\right) = \frac{\theta_1(x)}{2\sqrt[4]{q}\sin x},$$

some other products can be obtained via the identity $1 + q^n = \frac{1-q^{2n}}{1-q^n}$.

Miscellaneous products:

$$\prod_{n=0}^{\infty} \left(1 + x^{2^n}\right) = \frac{1}{1-x}.$$

Complex variables method:

$$\sum_{n\in\mathbb{Z}} (-1)^n f(n) = \frac{1}{2\mathrm{i}} \int_{\mathcal{C}} \frac{f(z)\,\mathrm{d}z}{\sin \pi z},$$

where the contour $\mathcal{C}$ encircles the whole real line and f has no singular points inside $\mathcal{C}$.

2.4. Continued fractions

Continued fraction is defined by

$$\mathrm{K}_{k=1}^{n}\left(a_k/b_k\right) = \frac{a_1}{b_1+}\,\frac{a_1}{b_2+}\cdots\frac{a_n}{b_n} = \cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots\cfrac{a_n}{b_n}}} \equiv \frac{A_n}{B_n},$$

where A_n and B_n are solutions of the three-term linear recurrence

$$X_n = b_n X_{n-1} + a_n X_{n-2}$$

with the initial conditions $A_0 = 0$, $A_1 = a_1$ and $B_0 = 1$, $B_1 = b_1$.

§3. Integrals

3.1. Indefinite integrals

Trigonometric:

$$\int R(\sin x, \cos x)\,\mathrm{d}x = \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{\mathrm{d}t}{1+t^2}, \quad t = \tan\frac{x}{2}.$$

Exponential and trigonometric:

$$\int x\begin{pmatrix}\sin ax\\ \cos ax\end{pmatrix}\mathrm{d}x = \frac{1}{a^2}\begin{pmatrix}\sin ax\\ \cos ax\end{pmatrix} \mp \frac{x}{a}\begin{pmatrix}\cos ax\\ \sin ax\end{pmatrix},$$

$$\int \mathrm{e}^{ax}\begin{pmatrix}\sin bx\\ \cos bx\end{pmatrix}\mathrm{d}x = \frac{\mathrm{e}^{ax}}{a^2+b^2}\left[a\begin{pmatrix}\sin bx\\ \cos bx\end{pmatrix} \mp b\begin{pmatrix}\cos bx\\ \sin bx\end{pmatrix}\right],$$

$$\int x^n \mathrm{e}^{ax}\,\mathrm{d}x = \mathrm{e}^{ax}\left(\frac{x^n}{a} - \frac{nx^{n-1}}{a^2} + \frac{n(n-1)x^{n-2}}{a^3} - \ldots\right).$$

Miscellaneous integrals:

$$\binom{n}{m} m \int_0^x y^{m-1}(1-y)^{n-m}\,\mathrm{d}y = \sum_{k=m}^{n}\binom{n}{k} x^k (1-x)^{n-k}.$$

3.2. Definite integrals

Exponential:

$$\int_{-\infty}^{x} \mathrm{e}^{-ax^2+bx}\,\mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}}\left(1+\mathrm{erf}\left(\sqrt{a}x - \frac{b}{2\sqrt{a}}\right)\right) \underset{x\to\infty}{\longrightarrow} \sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{b^2}{4a}},$$

$$\int_0^\infty x^{s\mu-1}\mathrm{e}^{-ax^s}\,\mathrm{d}x = \frac{\Gamma(\mu)}{sa^\mu}, \qquad \int_0^\infty x^{\mu-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x = \Gamma(\mu)\psi(\mu),$$

$$\int_0^\infty \mathrm{e}^{-ax^2+bx}\frac{\mathrm{d}x}{\sqrt{x}} = \pi\sqrt{\frac{|b|}{8a}}\,\mathrm{e}^{\frac{b^2}{8a}}\left[I_{-1/4}\left(\frac{b^2}{8a}\right) + \mathrm{sgn}\,q\,I_{1/4}\left(\frac{b^2}{8a}\right)\right].$$

Trigonometric:

$$\int_0^{\pi/2}\sin^\mu x\cos^\nu x\,\mathrm{d}x = \frac{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{\mu+\nu}{2}+1\right)}, \qquad \int_0^\pi \frac{\mathrm{d}x}{\sqrt{1-2k\cos x+k^2}} = 2\mathrm{K}(k),\ |k|<1,$$

$$\int_0^\pi \frac{\cos nx\,\mathrm{d}x}{(1-2a\cos x+a^2)^{m+1}} = \frac{\pi a^n}{(1-a^2)^{m+1}}\sum_{k=0}^{m}\binom{m+n}{k+n}\binom{m+k}{k}\left(\frac{a^2}{1-a^2}\right)^k,\ |a|<1.$$

Fourier:

$$\tanh a = 2\int_0^\infty \frac{\sin 2ax}{\sinh \pi x}\,\mathrm{d}x,$$
$$\coth a = \frac{1}{a} + 2\int_0^\infty \sin 2ax(\coth \pi x - 1)\,\mathrm{d}x,$$
$$\ln\sinh a = \ln a + \int_0^\infty (1-\cos 2ax)(\coth\pi x - 1)\frac{\mathrm{d}x}{x}.$$

3.3. Multiple integrals

The volume of a ball and the area of a sphere in $\mathbb{R}^n$ are given by

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}, \qquad S_n = nV_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

Miscellaneous integrals:

$$\int_{\mathbb{R}^3} r^{\mu-2}\exp\left(-ar + \mathrm{i}\boldsymbol{k}\boldsymbol{r}\right)\,\mathrm{d}V = \frac{4\pi\Gamma(\mu)\sin\mu\phi}{k\left(a^2+k^2\right)^{\mu/2}},\ \tan\phi = \frac{k}{a},$$
$$\int_{\mathbb{R}^n} \exp\left(-\alpha r^2 + \mathrm{i}\boldsymbol{k}\boldsymbol{r}\right)\,\mathrm{d}V = \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}}\exp\left(-\frac{k^2}{4\alpha}\right).$$

Integrals on quadratic form

Let consider integrals of the kind

$$I(x'') = \sqrt{\det A}\int_{\mathbb{R}^k} f\left(\sqrt{(x,Ax)};\ x\right)\,\mathrm{d}x',$$

where $x\in\mathbb{R}^n$ and $(x,Ax) = \sum_{i,j=1}^n A_{ij}x_ix_j$ with a symmetric positively defined matrix A. The $\mathbb{R}^k$ subspace is denoted by the prime and the $\mathbb{R}^{n-k}$ subspace is denoted by the double prime, so that $x = x' + x''$. Let

$$A = \begin{pmatrix} A' & M \\ M^\top & A'' \end{pmatrix}, \qquad A^{-1} = C = \begin{pmatrix} C' & N^\top \\ N & C'' \end{pmatrix},$$

so that $(x,Ax) = (x',A'x') + (x'',A''x'') + 2(x',Mx'')$. The integral I can be simplified by the diagonalization of $\mathbb{R}^k$ subspace produced by the substitution $x' = Sy - C'Mx''$, where $S^{-2} = A'$. The result is

$$I(x'') = \sqrt{\det\tilde{A}}\int_{\mathbb{R}^k} f\left(\sqrt{(y,y)+(x'',\tilde{A}x'')};\ Sy - C'Mx'' + x''\right)\,\mathrm{d}y,$$

where $\tilde{A} = (C'')^{-1}$. In particular,

$$\sqrt{\det A}\int_{\mathbb{R}^n} f\left(\sqrt{(x,Ax)},(a,x)\right)\,\mathrm{d}x = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\infty\int_0^\pi f(r,|Sa|r\cos\theta)r^{n-1}\sin^{n-2}\theta\,\mathrm{d}r\,\mathrm{d}\theta,$$
$$\sqrt{\det A}\int_{\mathbb{R}^n} f\left(\sqrt{(x,Ax)}\right)\,\mathrm{d}x = \frac{2\pi^{n/2}}{\Gamma(n/2)}\int_0^\infty f(r)r^{n-1}\,\mathrm{d}r,$$
$$\sqrt{\det A}\int_{\mathbb{R}^n} x_ix_jf\left(\sqrt{(x,Ax)}\right)\,\mathrm{d}x = C_{ij}\frac{2\pi^{n/2}}{n\Gamma(n/2)}\int_0^\infty f(r)r^{n+1}\,\mathrm{d}r.$$

Gauss integrals

Let us consider the following n-dimensional integral

$$I_k(a) = \frac{1}{\sqrt{\det(2\pi C)}}\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}(x,Ax)+(a,x)}\sum_{i_1\ldots i_k} B_{i_1i_2\ldots i_k}x_{i_1}x_{i_2}\ldots x_{i_k}\,\mathrm{d}x,$$

where the notations are the same as above. Substitution $x = y + Ca$ reduces this integral to

$$I_k(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)} \frac{1}{\sqrt{\det(2\pi C)}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}(y,Ay)} \sum_{i_1\dots i_k} B_{i_1\dots i_k} \left(y_{i_1} + (Ca)_{i_1}\right)\dots\left(y_{i_k} + (Ca)_{i_k}\right) \mathrm{d}y.$$

Integral $I_k(0)$ can be calculated explicitly: it will zero for odd k and

$$I_{2m}(0) = (2m-1)!! \sum_{i_1\dots i_{2m}} B_{[i_1 i_2\dots i_{2m}]} C_{i_1 i_2} C_{i_3 i_4}\dots C_{i_{2m-1}i_{2m}} \equiv (2m-1)!! \sum_{i_1\dots i_{2m}} B_{i_1 i_2\dots i_{2m}} C_{[i_1 i_2} C_{i_3 i_4}\dots C_{i_{2m-1}i_{2m}]},$$

where $[i_1 i_2 \dots i_{2m}]$ means the symmetrization over the all indexes, i.e.

$$B_{[i_1 i_2\dots i_{2m}]} \equiv \frac{1}{(2m)!} \sum_{j_1 j_2\dots j_{2m}=\mathrm{perm}(i_1 i_2\dots i_{2m})} B_{j_1 j_2\dots j_{2m}}.$$

For example,

$$\frac{1}{\sqrt{\det(2\pi C)}} \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{1}{2}(x,Ax)} x_i^{2m}\, \mathrm{d}V = (2m-1)!! C_{ii}^m.$$

For small k we have

$$I_0(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)}, \qquad I_1(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)}(B, Ca), \qquad I_2(a) = \mathrm{e}^{\frac{1}{2}(a,Ca)} \left[\mathrm{tr}(BC) + (a, CBCa)\right].$$

§4. Asymptotic methods

The directed equality $f(x) = O(\varphi(x)),\ x \to 0$ means $\exists a, c\ \forall x : |x| < a\ |f(x)| \leqslant c|\varphi(x)|$. If no limiting point is specified then the equivalence is considered as uniform. The directed equality $f(x) = o(\varphi(x)),\ x \to 0$ means $\lim_{x\to 0} f(x)/\varphi(x) = 0$. The equality $f(x) \sim \varphi(x),\ x \to 0$ means $\lim_{x\to 0} f(x)/\varphi(x) = 1$. The asymptotic expansion $f(x) \sim \sum_{k=1}^{\infty} c_k \varphi_k(x),\ x \to 0$ means $f(x) - \sum_{k=1}^{n} c_k\varphi_k(x) = o(\varphi_n(x)),\ x \to 0$. These relations are algebraically transitive and admit integration, differentiation is generally not allowed.

4.1. Summation

Sums with smoothly varying terms are evaluated by Euler–Maclaurin formula:

$$\sum_{k=0}^{n} f(hk) = \frac{1}{h}\int_0^{hn} f(x)\,\mathrm{d}x + \frac{f(0)+f(hn)}{2} - \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(0) - f^{(2k-1)}(hn)\right] h^{2k-1} - h^{2m}\int_0^n \frac{\tilde{B}_{2m}(y)}{(2m)!} f^{(2m)}(hy)\,\mathrm{d}y,$$

where

$$\frac{\tilde{B}_{2m}(y)}{(2m)!} = 2(-1)^{m+1} \sum_{k=1}^{\infty} \frac{\cos 2\pi k y}{(2\pi k)^{2m}}.$$

For sums with oscillating terms there are following methods. In case of alternating terms the sum can be converted to the one with positive terms by abelian transformation:

$$\sum_{k=0}^{n} a_k b_k = A_n b_n + \sum_{k=0}^{n-1} A_k (b_k - b_{k+1}), \qquad A_n = \sum_{k=0}^{n} a_k.$$

In other cases one can use Poisson's summation formula:

$$\sum_{k=-\infty}^{\infty} f(k+a) = \lim_{n\to\infty} \sum_{k=-n}^{n} \mathrm{e}^{2\pi\mathrm{i}ka} \int_{-\infty}^{\infty} \mathrm{e}^{-2\pi\mathrm{i}kx} f(x)\,\mathrm{d}x,$$

providing that the sums $\sum_k f(k+x)$ and $\sum_k f'(k+x)$ converge uniformly for $0 \leqslant x \leqslant 1$.

See also abelian and tauberian theorems in Section 6.3.

4.2. Integration

See abelian and tauberian theorems in Section 5.2.

4.3. Saddle-point methods

Here we consider asymptotics of integral $\int \mathrm{e}^{\lambda S(z)} f(z)\, \mathrm{d}z$ as $\lambda \to +\infty$.

Let x_0 be a nondegenerate global maximum of S and let x_0 be an inner point of the integration interval then

$$\int \mathrm{e}^{\lambda S(x)} f(x)\, \mathrm{d}x \sim \mathrm{e}^{\lambda S(x_0)} \sqrt{\frac{2\pi}{-\lambda S''(x_0)}} \left[\sum_{k=0}^{3n} c_{2k}(\lambda) \left(\frac{1}{2}\right)_k \left(\frac{2}{-\lambda S''(x_0)}\right)^k + o\left(\lambda^{-n}\right) \right], \tag{4.1}$$

where c_m are defined via the following generating function

$$f(x) \exp \lambda \left[S(x) - S(x_0) - \frac{1}{2} S''(x_0)(x - x_0)^2 \right] = \sum_{m=0}^{\infty} c_m(\lambda)(x - x_0)^m.$$

If x_0 coincides with the boundary point of the integration interval then one must divide the right-hand side of (4.1) by 2. The first two terms in (4.1) can be written explicitly:

$$\int \mathrm{e}^{\lambda S(x)} f(x)\, \mathrm{d}x \sim \mathrm{e}^{\lambda S(x_0)} \sqrt{\frac{2\pi}{-\lambda S''(x_0)}} \left[f + \frac{1}{\lambda} \left(-\frac{f''}{2S''} + \frac{f S''''}{8 S''^2} + \frac{f' S'''}{2 S''^2} - \frac{5 f S'''^2}{24 S''^3} \right) + \dots \right]_{x=x_0}.$$

In a multidimensional case

$$\int \mathrm{e}^{\lambda S(x)} f(x)\, \mathrm{d}x \sim \frac{\mathrm{e}^{\lambda S(x_0)}}{\sqrt{\det \frac{\lambda Q}{2\pi}}} \sum_{k=0}^{\infty} \frac{1}{k! 2^k \lambda^k} \left(Q^{-1}_{ij} \partial^i \partial^j \right)^k \left\{ f(x) \exp \lambda \left[S(x) - S(x_0) - \frac{1}{2} Q_{ij}(x^i - x^i_0)(x^j - x^j_0) \right] \right\} \Big|_{x=x_0}.$$

where $Q_{ij} = -\left.\frac{\partial^2 S}{\partial x^i \partial x^j}\right|_{x_0}$ and the summation over the dummy indexes is used. The leading term in the above series is $f(x_0)$.

If x_0 is a degenerate maximum such that $S(x) - S(x_0) \sim (x - x_0)^{2m}$ then the leading term is

$$\int \mathrm{e}^{\lambda S(x)} f(x)\, \mathrm{d}x \sim \frac{1}{m} \Gamma\left(\frac{1}{2m}\right) \mathrm{e}^{\lambda S(x_0)} \sqrt[2m]{-\frac{(2m)!}{\lambda S^{(2m)}(x_0)}} f(x_0).$$

If the global maximum of S is reached at the left boundary of the integration interval a and $S'(a) \neq 0$ then

$$\int_a \mathrm{e}^{\lambda S(x)} f(x)\, \mathrm{d}x \sim \mathrm{e}^{\lambda S(a)} \sum_{k=0}^{\infty} \lambda^{-k-1} \left(-\frac{\mathrm{d}}{S'(x)\, \mathrm{d}x} \right)^k \left(-\frac{f(x)}{S'(x)} \right) \Big|_{x=a} = -\frac{\mathrm{e}^{\lambda S(a)}}{\lambda} \frac{f(a)}{S'(a)} + \dots$$

Some examples of oscillating integrals:

$$\int_0^{\pi-\varepsilon} \mathrm{e}^{\mathrm{i}\lambda \cos\phi} f(\phi)\, \mathrm{d}\phi \sim \sqrt{\frac{\pi}{2\lambda}} \mathrm{e}^{\mathrm{i}(\lambda - \pi/4)} f(0).$$

See also [2].

§5. Integral transforms

5.1. Fourier transform

Fourier transform is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{d}x, \;\; k \in \mathbb{C},$$

so that $\nabla \to \mathrm{i}k$ as in quantum mechanics. Its inverse is given by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k,\ x \in \mathbb{R}.$$

In mathematical literature the symmetric form is used, and the second integral is considered as principal value. Besides (x, k) pair, in physical literature (t, ω) pair is used, but with opposite sign:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t,$$

to perform Fourier transform by plane waves $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$.

Some identities:

$$\int_{-\infty}^{\infty} \overline{f(x)}g(x)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(k)}\hat{g}(k)\,\mathrm{d}k,$$
$$\int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y \to \hat{f}(k)\hat{g}(k).$$

Transformation table (n is the dimension):

$$\delta(x) \to 1,$$
$$\exp\left(-\alpha r^2\right) \to \left(\frac{\pi}{\alpha}\right)^{\frac{n}{2}} \exp\left(-\frac{k^2}{4\alpha}\right),$$
$$\mathrm{e}^{-\alpha r} \to \frac{2^n \pi^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)\alpha}{(\alpha^2+k^2)^{\frac{n+1}{2}}},$$
$$\frac{r^{m-\frac{n}{2}}K_{m-\frac{n}{2}}(\alpha r)}{2^{m+\frac{n}{2}-1}\pi^{\frac{n}{2}}\Gamma(m)\alpha^{m-\frac{n}{2}}} \to \frac{1}{(\alpha^2+k^2)^m}$$

n-dimensional Fourier transform of axially symmetric function $f(\boldsymbol{x}) = f(r, \theta)$ can be calculated by formulas

$$\hat{f}(k,\beta) = S_{n-2}\int_0^{\infty} r^{n-1}\,\mathrm{d}r \int_0^{\pi} f(r,\theta)\mathrm{e}^{-\mathrm{i}kr\cos\theta}\sin^{n-2}\theta\,\mathrm{d}\theta,$$
$$f(r,\theta) = \frac{S_{n-2}}{(2\pi)^n}\int_0^{\infty} k^{n-1}\,\mathrm{d}k \int_0^{\pi} \hat{f}(k,\beta)\mathrm{e}^{\mathrm{i}kr\cos\beta}\sin^{n-2}\beta\,\mathrm{d}\beta.$$

In the case of spherical symmetry

$$\hat{f}(k) = \frac{(2\pi)^{n/2}}{k^{\frac{n}{2}-1}}\int_0^{\infty} f(r)J_{\frac{n}{2}-1}(kr)r^{n/2}\,\mathrm{d}r, \qquad f(r) = \frac{1}{(2\pi)^{n/2}r^{\frac{n}{2}-1}}\int_0^{\infty} \hat{f}(k)J_{\frac{n}{2}-1}(kr)k^{n/2}\,\mathrm{d}k.$$

5.2. Laplace transform

Laplace transform is defined by

$$\tilde{f}(s) = \int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-st}\,\mathrm{d}F(t),\ s \in \mathbb{C},$$

where the second expression is a more general form with monotonic non-decreasing F (if F is differentiable then $f \equiv F'$). The inverse transformation is given by

$$f(t) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \tilde{f}(s)\mathrm{e}^{st}\,\mathrm{d}s,\ t \in \mathbb{R}_+.$$

The function $f(t)$ is assumed to be zero for $t < 0$, $\tilde{f}(s)$ must be analytic for $\Re s > \sigma$. If the only singularities of $\tilde{f}$ are branch cut $(-p, 0)$ and isolated singular points then one can use the following formula in the limit $r \to 0$:

$$f(t) = \sum_s \operatorname{res} \tilde{f}(s)\mathrm{e}^{st} + \frac{1}{2\pi\mathrm{i}}\int_r^{p-r}\left[\tilde{f}(-x-\mathrm{i}0) - \tilde{f}(-x+\mathrm{i}0)\right]\mathrm{e}^{-xt}\,\mathrm{d}x$$
$$+ \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(r\mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\phi}} r\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi + \frac{\mathrm{e}^{-pt}}{2\pi}\int_0^{2\pi} \tilde{f}(-p+r\mathrm{e}^{\mathrm{i}\phi})\mathrm{e}^{tr\mathrm{e}^{\mathrm{i}\phi}} r\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{d}\phi.$$

Some properties:

- shift and dilatation:

$$f(t-a) \to \mathrm{e}^{-as}\tilde{f}(s),\ a \geqslant 0, \qquad \mathrm{e}^{at}f(t) \to \tilde{f}(s-a), \qquad f(at) \to \frac{1}{a}\tilde{f}\left(\frac{s}{a}\right),$$

- differentiation and integration:

$$f^{(n)}(t) \to s^n\tilde{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0) = \frac{1}{s}\operatorname*{Regular}_{s\to\infty}\left[s^{n+1}\tilde{f}(s)\right],$$

$$t^n f(t) \to (-1)^n\tilde{f}^{(n)}(s), \qquad \int_0^t f(\tau)\,\mathrm{d}\tau \to \frac{1}{s}\tilde{f}(s), \qquad \frac{1}{t}f(t) \to \int_s^\infty \tilde{f}(\sigma)\,\mathrm{d}\sigma,$$

- convolution and product:

$$\int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau \to \tilde{f}(s)\tilde{g}(s), \qquad f(t)g(t) \to \frac{1}{2\pi\mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty}\tilde{f}(\sigma)g(s-\sigma)\,\mathrm{d}\sigma,$$

- special properties:

$$\int_0^\infty f(\tau)g(t,\tau)\,\mathrm{d}\tau \to \tilde{f}(q(s))\tilde{g}(s),\ \text{where } g(t,\tau) \to \mathrm{e}^{-\tau q(s)}\tilde{g}(s),$$

$$\forall t \geqslant 0\ f(t) \geqslant g(t) \implies \forall s \geqslant 0\ \tilde{f}(s) \geqslant \tilde{g}(s).$$

Transformation table:

$$\mathrm{e}^{at} \to \frac{1}{(s-a)}, \qquad \cos at + \mathrm{i}\sin at \to \frac{s+\mathrm{i}a}{s^2+a^2}, \qquad t^{\mu-1} \to \frac{\Gamma(\mu)}{s^\mu}, \qquad t^{\mu-1}\left[\psi(\mu) - \ln t\right] \to \frac{\Gamma(\mu)}{s^\mu}\ln s,$$

$$\frac{1}{(t+a)^{\mu+1}} \to \mathrm{e}^{as}s^\mu\Gamma(-\mu, as),\ a>0, \qquad \gamma(\mu, at)\mathrm{e}^{at} \to \frac{\Gamma(\mu)a^\mu}{s^\mu(s-a)}, \qquad J_\mu(at) \to \frac{\left(\sqrt{s^2+a^2}-s\right)^\mu}{a^\mu\sqrt{s^2+a^2}},$$

$$\mathrm{e}^{-a\sqrt{t}} \to \frac{1}{s}\left[1 - \frac{a\sqrt{\pi}}{2\sqrt{s}}\mathrm{e}^{\frac{a^2}{4s}}\left(1 - \operatorname{erf}\left(\frac{a}{2\sqrt{s}}\right)\right)\right], \qquad t^{\mu-1}\mathrm{e}^{-\frac{a}{t}} \to 2\left(\frac{a}{s}\right)^{\frac{\mu}{2}}K_\mu(2\sqrt{as}),$$

$$-\mathrm{Ei}\left(-\frac{t}{a}\right) \to \frac{\ln(1+as)}{s}, \qquad \mathrm{e}^t\mathrm{Li}_\mu\left(-\mathrm{e}^t\right) \to \frac{\pi}{(1+s)^\mu\sin\pi s}.$$

Regular series

Series expansion around $s=\infty$:

$$\tilde{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \ldots + \frac{f^{(n)}(0)}{s^{n+1}} + \ldots$$

Expansion in Laguerre polynomials can provide numerical inversion:

$$f(t) = t^a\sum_{n=0}^\infty c_n L_n^a(t),\ \text{where } c_n = \frac{(-1)^n}{\Gamma(n+a+1)}\left.\frac{\mathrm{d}^n}{\mathrm{d}s^n}\left(\frac{1}{s^{a+1}}\tilde{f}\left(\frac{1}{s}\right)\right)\right|_{s=1}.$$

Asymptotic expansion as $t\to\infty$

By argument shift one can move any singularity to $s=0$. If $s=0$ is an isolated singular point then one can use the residue formula. In the case of non-isolated singularities we obtain asymptotic expansion as $t\to\infty$. In particular, let $s=0$ be an algebraic branch point and thus

$$\tilde{f}(s) = s^\mu\sum_{n=0}^\infty c_n s^n$$

or linear combination of such generalized series. Then for $f(t)$ we obtain the following asymptotic expansion:

$$f(t) \sim -\frac{\sin\pi\mu}{\pi}\frac{1}{t^{\mu+1}}\sum_{n=0}^\infty\frac{(-1)^n c_n\Gamma(n+\mu+1)}{t^n}.$$

For simple logarithmic singularity one have similarly

$$\tilde{f}(s) = \ln s \sum_{n=0}^{\infty} c_n s^n \implies f(t) \sim -\frac{1}{t} \sum_{n=0}^{\infty} \frac{(-1)^n c_n \Gamma(n+1)}{t^n}.$$

For complex logarithmic singularities one can use the correspondence

$$s^n \ln^m s \leftarrow \frac{(-1)^m m!}{t^{n+1}} \sum_{k=0}^{\min(m-1,n)} S_{n+1}^{k+1} \left(\frac{t^z}{\Gamma(z)} \right)_{m-k},$$

where S_n^k are Stirling's numbers of the first kind and $()_k$ is the k-th term in series expansion at $z=0$, which can be obtained from

$$\frac{t^z}{\Gamma(z)} = \frac{\sin \pi z}{\pi} \exp \left[(\gamma + \ln t)\, z + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} z^k \right].$$

Tauberian theorems

Let call function φ slow varying if $\forall \lambda > 0\ \lim_{x\to\infty} \frac{\varphi(\lambda x)}{\varphi(x)} = 1$ (logarithmic function and any function having a limit at infinity are examples). It can be shown [1, v.2, p.508] that

$$\tilde{f}(s) \sim s^{-\mu} \varphi(1/s),\ s \to +0 \iff F(t) \sim \frac{t^\mu}{\Gamma(\mu+1)} \varphi(t), \quad t \to \infty,$$
$$\tilde{f}(s) \sim s^{-\mu} \varphi(s), \quad s \to \infty \iff F(t) = \frac{t^\mu}{\Gamma(\mu+1)} \varphi(1/t),\ t \to +0.$$

If F is differentiable then

$$s\tilde{f}(s) \sim s^{-\mu} \varphi(1/s),\ s \to +0 \iff f(t) \sim \frac{t^\mu}{\Gamma(\mu+1)} \varphi(t),\ t \to \infty,$$

and $\lim_{s\to 0} s\tilde{f}(s) = \langle f \rangle_{t\to\infty}$ if the limit exists.

Laplace transforms of logarithmic functions

One can show that

$$t^{\nu-1} \ln^m t \to \frac{\Gamma(\nu)}{s^\nu} P_m^\nu(\ln s) \qquad \text{and} \qquad \frac{1}{s^\nu} \ln^m s \leftarrow \frac{t^{\nu-1}}{\Gamma(\nu)} Q_m^\nu(\ln t),$$

where $m \in \mathbb{Z}_+$, P_m^ν and Q_m^ν are polynomials of the order m. Explicit formulas for these polynomials are

$$P_m^\nu(x) = \frac{\mathrm{e}^{\nu x}}{\Gamma(\nu)} \frac{\partial^m (\mathrm{e}^{-\nu x} \Gamma(\nu))}{\partial \nu^m}, \qquad Q_m^\nu(x) = -\Gamma(\nu) \mathrm{e}^{-\nu x} \frac{\partial^m}{\partial \nu^m} \left(\frac{\mathrm{e}^{\nu x}}{\Gamma(\nu)} \right).$$

They can be found with the use of the recurrence relations

$$P_0^\nu(x) = +1, \qquad P_1^\nu(x) = \psi(\nu) - x, \qquad P_{m+1}^\nu(x) = \frac{\partial P_m^\nu(x)}{\partial \nu} + P_1^\nu(x) P_m^\nu(x),$$
$$Q_0^\nu(x) = -1, \qquad Q_1^\nu(x) = \psi(\nu) - x, \qquad Q_{m+1}^\nu(x) = \frac{\partial Q_m^\nu(x)}{\partial \nu} - Q_1^\nu(x) Q_m^\nu(x).$$

Note that the difference between P and Q is only in opposite signs at $\psi^{(2k+1)}(\nu),\ k \in \mathbb{Z}_+$. If $\nu \leqslant 0$ then the transform is undefined. But we can easily define inverse Laplace transforms for these values of ν by the above formula, taking limit for non-positive integer ν. Direct Laplace transform is still undefined for non-positive integer ν by the above formula, but it can be defined via the inverse transform in this case.

§6. Discrete transforms

6.1. Fourier series

Fourier series is defined in context of the following transformation:

$$\hat{f}(k) = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}kn},\ k \in (-\pi, \pi],$$
$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{f}(k)\mathrm{e}^{-\mathrm{i}kn}\,\mathrm{d}k,\ n \in \mathbb{Z}.$$

$\hat{f}(k)$ is regarded as 2π-periodic so one can choose $k \in [0, 2\pi)$ also.

Some properties:

- transform of unit function:
$$1 \to 2\pi\delta(k), \qquad \delta_{n0} \to 1,$$
- argument shift and inversion:
$$f_{n+a} \to \hat{f}(k)\mathrm{e}^{-\mathrm{i}ka}, \qquad f_n\mathrm{e}^{\mathrm{i}pn} \to \hat{f}(k+p), \qquad f_{-n} \to \hat{f}(-k) \equiv \hat{f}(2\pi - k),$$
- multiplication on argument and differentiation:
$$nf_n \to -\mathrm{i}\frac{\mathrm{d}\hat{f}(k)}{\mathrm{d}k}, \qquad f_n - f_{n-1} \to \hat{f}(k)(1-\mathrm{e}^{\mathrm{i}k}),$$
- product and convolution:
$$f_n g_n \to \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(\kappa)\hat{g}(k-\kappa)\,\mathrm{d}\kappa, \qquad \sum_{m\in\mathbb{Z}} f_m g_{n-m} \to \hat{f}(k)\hat{g}(k),$$
- other identities:
$$\sum_{n\in\mathbb{Z}} f_n g_n = \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(k)\hat{g}(2\pi-k)\,\mathrm{d}k, \qquad \sum_{n\in\mathbb{Z}} f_n g_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \hat{f}(k)\hat{g}(k)\,\mathrm{d}k.$$

Transformation table:

$$\mathrm{e}^{-\alpha|n|} \to \frac{1-q^2}{1-2q\cos k + q^2}.$$

6.2. Discrete Fourier transform

Discrete Fourier transform is defined by

$$\hat{f}_k = \sum_{n=0}^{L-1} f_n \mathrm{e}^{\mathrm{i}kn},\ k = \frac{2\pi}{L}l,\ l = \overline{0, L-1,},$$
$$f_n = \frac{1}{L}\sum_k \hat{f}_k \mathrm{e}^{-\mathrm{i}kn},\ n = \overline{0, L-1,},$$

both f_n and $\hat{f}_k$ are L-periodic. If $\hat{f}_k$ is even function then

$$f_n = \frac{1}{L}\hat{f}_0 + \frac{2}{L}\sum_{l=1}^{\left[\frac{L-1}{2}\right]} \hat{f}_l \cos\left(\frac{2\pi nl}{L}\right) + \mathcal{I}\left\{L \text{ is even}\right\}\frac{(-1)^n}{L}\hat{f}_{\frac{L}{2}}.$$

Some properties:

- transform of unit function:

$$1 \to L\delta_{k0}, \qquad \delta_{n0} \to 1,$$

- argument shift:

$$f_{n+a} \to \hat{f}_k \mathrm{e}^{-\mathrm{i}ka}, \qquad f_n \mathrm{e}^{\mathrm{i}pn} \to \hat{f}_{k+p},$$

- discrete differentiation:

$$f_n - f_{n-1} \to \hat{f}(k)(1 - \mathrm{e}^{\mathrm{i}k}),$$

- product and convolution:

$$f_n g_n \to \frac{1}{L} \sum_{\kappa} \hat{f}_\kappa \hat{g}_{k-\kappa}, \qquad \sum_{m=0}^{L-1} f_m g_{n-m} \to \hat{f}_k \hat{g}_k,$$

- other identities:

$$\sum_{n=0}^{L-1} f_n g_n = \frac{1}{L} \sum_{k} \hat{f}_k \hat{g}_{2\pi-k}, \qquad \sum_{n=0}^{L-1} f_n g_{L-n} = \frac{1}{L} \sum_{k} \hat{f}_k \hat{g}_k.$$

Transformation table:

$$\sum_{k\neq 0} \frac{1}{1-\cos k} = \frac{L^2-1}{6}.$$

6.3. Generating function

Generating function is defined in context of the following transformation:

$$\tilde{f}(q) = \sum_{n=0}^{\infty} f_n q^n, \ q \in \mathbb{C},$$

$$f_n = \frac{1}{2\pi \mathrm{i} n!} \oint_{q=0} \frac{\tilde{f}(q)\,\mathrm{d}q}{q^{n+1}} \equiv \frac{\tilde{f}^{(n)}(0)}{n!}, \ n \in \mathbb{Z}_+,$$

here $\tilde{f}(q)$ is the generating function of the sequence $\{f_n\}$; f_n is assumed to be zero for $n < 0$.

Some properties:

- transform of unit function:

$$1 \to \frac{1}{1-q}, \qquad \delta_{n0} \to q^n,$$

- shift and dilatation:

$$f_{n-a} \to \tilde{f}(q) q^a, \ a \geqslant 0, \qquad f_n p^n \to \hat{f}(pq),$$

- multiplication on argument, differentiation, and summation:

$$n f_n \to q \frac{\mathrm{d}}{\mathrm{d}q} \tilde{f}(q), \qquad f_{n+1} - f_n \to \frac{\tilde{f}(q)(1-q) - f_0}{q}, \qquad f_n - f_{n-1} \to \tilde{f}(q)(1-q), \qquad \sum_{m=0}^{n} f_m \to \frac{\tilde{f}(q)}{1-q},$$

- convolution:

$$\sum_{m=0}^{n} f_m g_{n-m} \to \tilde{f}(q) \tilde{g}(q).$$

Tauberian theorems: Karamata proved for slow varying function L that

$$\tilde{f}(q) \sim \frac{1}{(1-q)^\mu} L\left(\frac{1}{1-q}\right), \ q \to 1-0, \ \mu \geqslant 0 \implies \sum_{m=0}^{n} f_m \sim \frac{n^\mu}{\Gamma(\mu+1)} L(n), \ n \to \infty,$$

and if the sequence $\{f_n\}$ is monotonic then

$$f_n \sim \frac{n^{\mu-1}}{\Gamma(\mu)} L(n), \ n \to \infty.$$

In particular, if $\mu = 0$ then the statement becomes trivial: $\sum_{m=0}^{\infty} f_m = \tilde{f}(1)$.

References

[1] W Feller, An introduction to probability theory and its applications (New York, Wiley, 1957, 1966); cited is russian edition of 1984.

[2] M V Fedoriuk, Asymptotics: Integrals and series (Moscow, Nauka, 1987); in russian.